Discrete Mathematics, KOM1062
Lecture #2

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<u>Lecture Book:</u> **"Discrete Mathematics, Seventh Edt., Kenneth H. Rosen, 2007, McGraw Books Discrete Mathematics and Applications, Susanna S. Epp, Brooks, 4th Edt., 2011".**

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Propositional Equivalences

- An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value
- Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

<u>**Definition 1:**</u> A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology. A compound proposition that is always* false is called a *contradiction. A compound proposition that is neither a tautology nor a* contradiction is called a *contingency.*

Ex. 1(25): We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ *and* $p \wedge \neg p$*, shown in the Table. Because* $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ *is always false, it is a contradiction.*

✓ Compound propositions that have the same truth values in all possible cases are called **logically equivalent.**

<u>**Definition 2:**</u> The compound propositions *p and q are called logically equivalent if* $p \leftrightarrow q$ *is a tautology.* The notation $p \equiv q$ *denotes that p and q are logically equivalent.*

Note that, The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

Example for the Tautology and Contradiction Cases

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

- One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions *p and q are equivalent if and only if the columns* giving their truth values agree. Following example illustrates this method to establish an extremely important and useful logical equivalence, namely, that of $\neg(p \vee q)$ *with* $\neg p \wedge \neg q$. *This logical* equivalence is one of the two De Morgan laws, shown in the Table, named after the English mathematician Augustus De Morgan, of the mid-nineteenth century.

De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Ex. 2(26): Show that $\neg(p \vee q)$ *and* $\neg p \wedge \neg q$ *are logically equivalent.*
Solution:

Truth Table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Ex. 3: Show that $p \rightarrow q$ *and* $\neg p \vee q$ *are logically equivalent.*
Solution:

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Ex. 4: Show that $p \vee (q \wedge r)$ *and* $(p \vee q) \wedge (p \vee r)$ *are logically equivalent. This is the distributive law of disjunction over conjunction.*
Solution:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

- Following table contains some important equivalences. In these equivalences, **T denotes the compound** proposition that is always true and **F denotes the compound proposition that is always** false. We also display some useful equivalences for compound propositions involving conditional statements and biconditional statements in the tables given at the next slight.

Logical Equivalant Propositions	
$p \wedge T \equiv p$ $p \vee F \equiv p$	İdentity Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Law
$p \vee \neg p = T$ $p \wedge \neg p = F$	Negation Law

- The associative law for disjunction shows that the expression $p \vee q \vee r$ *is well defined,* in the sense that it does not matter whether we first take the disjunction of p *with* q *and then* the disjunction of $p \vee q$ *with* r*, or if we first take the disjunction of* q *and* r *and then take the* disjunction of p *with* $q \vee r$*. Similarly, the expression* $p \wedge q \wedge r$ *is well defined. By extending this* reasoning, it follows that $p_1 \vee p_2 \vee \cdots \vee p_n$ *and* $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ *are well defined whenever* $p_1, p_2, \ldots, p_n$ *are propositions.*

- Furthermore, note that De Morgan's laws extend to

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$
$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n).$$

Logical Equivalences of Some Conditional Statements
$p \to q \equiv \neg p \vee q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (p \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences of Some Biconditional Statements
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

De Morgan's Laws: The two logical equivalences known as De Morgan's laws are particularly important. They tell us how to negate conjunctions and how to negate disjunctions. In particular, the equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ *tells us that the negation of a disjunction is formed by taking the conjunction* of the negations of the component propositions. Similarly, the equivalence $\neg(p \wedge q) \equiv \neg p \vee \neg q$ *tells us that the negation of a conjunction is formed by taking the disjunction of the* negations of the component propositions.

Ex. 5: Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution: Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone or he does not have a laptop computer."

Constructing New Logical Equivalences: The logical equivalences as well as any others that have been established, can be used to construct additional logical equivalences. The reason for this is that a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.

Ex. 6: Show that *$\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.*

$$\begin{aligned}\neg(p \to q) &\equiv \neg(\neg p \vee q)\\ &\equiv \neg(\neg p) \wedge \neg q\\ &\equiv p \wedge \neg q\end{aligned}$$

Ex. 7: Show that *$\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of* logical equivalences.

Solution:

$$\begin{aligned}\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law}\\ &\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law}\\ &\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law}\\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law}\\ &\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F}\\ &\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction}\\ &\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}\end{aligned}$$

Ex. 8: Show that *$(p \wedge q) \to (p \vee q)$ is a tautology.*

Solution:

$$\begin{aligned}(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q)\\ &\equiv (\neg p \vee \neg q) \vee (p \vee q)\\ &\equiv (\neg p \vee p) \vee (\neg q \vee q)\\ &\equiv \mathbf{T} \vee \mathbf{T}\\ &\equiv \mathbf{T}\end{aligned}$$

Propositional Satisfiability: A compound proposition is **satisfiable if there is an assignment of truth values to its variables that** makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable.**

Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

Ex. 9: Determine whether each of the compound propositions $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$, and $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ *is satisfiable.*

Solution: ***S/not-S***

Predicates and Quantifiers

- We will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects.
- To understand predicate logic, we first need to introduce the concept of a predicate. Afterward, we will introduce the notion of quantifiers, which enable us to reason with statements that assert that a certain property holds for all objects of a certain type and with statements that assert the existence of an object with a particular property.

Predicates: Statements involving variables, such as

"x > 3," "x = y + 3," "x + y = z,"

"computer *x is under attack by an intruder,"*

"computer *x is functioning properly,"*

called as predicate statements. These statements are neither true nor false when the values of the variables are not specified. But they enable us to produce another statements.

Ex. 1(37): Let *P(x) denote the statement "x > 3." What are the truth values of P(4) and P(2)?*

Solution:

Ex. 3: Let *Q(x, y) denote the statement "x = y + 3." What* are the truth values of the propositions *Q(1, 2) and Q(3, 0)?*

Solution:F/T

Ex. 4: Let *A(c, n) denote the statement "Computer c is connected to network n," where c is a variable* representing a computer and *n is a variable representing a network. Suppose that the computer* MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of *A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?*

Solution: F/T

Ex. 5: let *R(x, y, z) denote the statement x + y = z.* What are the truth values of the propositions *R(1, 2, 3) and R(0, 0, 1)?*

Solution:

Ex. 6: Consider the statement

if *x > 0 then x := x + 1.*

Solution:

PRECONDITIONS AND POSTCONDITIONS:

- Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input.
- The statements that describe valid input are known as preconditions and the conditions that the output should satisfy when the program has run are known as postconditions.

Ex. 7(39): Consider the following program, designed to interchange the values of two variables *x and y.*

```
temp := x
x := y
y := temp
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Solution:

Quantifiers: When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value.

- However, there is another important way, called quantification, to create a proposition from a propositional function.
- Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all, some, many, none, and few are used in quantifications.*
- *We will focus on two types of quantification* here: universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called the predicate calculus.

The Universal Quantifier: Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the domain of discourse (or the universe of discourse), often just referred to as the domain.

- Such a statement is expressed using universal quantification. The universal quantification of *P(x) for a particular domain is the* proposition that asserts that *P(x) is true for all values of x in this domain.(*The domain must always be specified when a universal quantifier is used*)*

Definition 1: The *universal quantification of P(x) is the statement*

"P(x) for all values of x in the domain."

The notation ∀*xP(x) denotes the universal quantification of P(x). Here* ∀ *is called the* universal quantifier. We read ∀*xP(x) as "for all x P(x)" or "for every x P(x)." An element* for which *P(x) is false is called a Counter example of* ∀*xP(x).*

Quantifiers
∀x P(x): True, if P(x) is true for every x; False, if there is an x for which P(x) is false.
∃x P(x): True, if there is an x for which P(x) is true; False, if P(x) is false for every x.

Ex. 8: Let *P(x) be the statement "x + 1 > x." What is the truth value of the quantification* ∀*xP(x),* where the domain consists of all real numbers?
Solution: ∀*xP(x) is true for real number set.*

✓A statement ∀*xP(x) is false, where P(x) is a propositional function, if and only if P(x) is not* always true when *x is in the domain.*

Ex. 9: Let *Q(x) be the statement "x < 2." What is the truth value of the quantification* ∀*xQ(x), where* the domain consists of all real numbers?
Solution: Counterexample : x=3.

Ex. 10: Suppose that *P(x) is "$x^2 > 0$." To show that the statement* ∀*xP(x) is false where the universe* of discourse consists of all integers, we give a counter example. We see that *x = 0 is a* Counter example because $x^2 = 0$ *when x = 0, so that* x^2 *is not greater than 0 when x = 0.*

✓When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—*it follows that the* universal quantification ∀*xP(x) is the same as the conjunction*

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ *are all true.*

Ex. 11: What is the truth value of ∀*xP(x), where P(x) is the statement "$x^2 < 10$" and the domain* consists of the positive integers not exceeding 4?
Solution:

$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ "$4^2 < 10$," is false, it follows that $\forall x\, P(x)$ is false.

Ex. 13: What is the truth value of ∀*x($x^2 \geq x$) if the domain consists of all real numbers? What is the* truth value of this statement if the domain consists of all integers?
Solution: False

$(\frac{1}{2})^2 \ngeq \frac{1}{2}$

THE EXISTENTIAL QUANTIFIER: With existential quantification, we form a proposition that is true if and only if *P(x) is true for* at least one value of *x in the domain.* A domain must always be specified when a statement ∃*xP(x) is used.*

Definition 1: The *existential quantification of P(x) is the proposition*

"There exists an element *x in the domain such that P(x)."*

We use the notation ∃*xP(x) for the existential quantification of P(x). Here* ∃ *is called the existential quantifier.*

Ex. 14: Let *P(x) denote the statement "$x > 3$." What is the truth value of the quantification $\exists xP(x)$,* where the domain consists of all real numbers?
Solution: True

Ex. 15: Let *Q(x) denote the statement "$x = x + 1$."What is the truth value of the quantification $\exists xQ(x)$,* where the domain consists of all real numbers?
Solution: False

✓If the domain is empty, then $\exists xQ(x)$ *is false whenever Q(x) is a propositional* function because when the domain is empty, there can be no element *x in the domain for which Q(x) is true.*

✓When all elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—*the existential quantification* $\exists xP(x)$ *is the same as the disjunction*

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ *is true.*

Ex. 16: What is the truth value of $\exists xP(x)$, *where P(x) is the statement "$x^2 > 10$" and the universe of* discourse consists of the positive integers not exceeding 4?
Solution:

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true.

THE UNIQUENESS QUANTIFIER: These are the most important quantifiers in mathematics and computer science. However, there is no limitation on the number of different quantifiers we can define, such as "there are exactly two," "there are no more than three," "there are at least 100," and so on. Of these other quantifiers, the one that is most often seen is the **uniqueness quantifier, denoted by ∃! or ∃1.** The notation $\exists! xP(x)$ *[or $\exists 1xP(x)$] states "There exists a unique x such that P(x) is true."*

- For instance, $\exists! x(x - 1 = 0)$, *where the domain is the set of real numbers, states* that there is a unique real number *x such that $x - 1 = 0$. This is a true statement, as $x = 1$ is the* unique real number such that $x - 1 = 0$.

Precedence of Quantifiers: The quantifiers ∀ and ∃ have higher precedence than all logical operators from propositional calculus. For example, $\forall xP(x) \vee Q(x)$ *is the disjunction of $\forall xP(x)$ and Q(x). In other words,* it means $(\forall xP(x)) \vee Q(x)$ *rather than* $\forall x(P(x) \vee Q(x))$.

Binding Variables: When a quantifier is used on the variable *x, we say that this occurrence of the variable is* ***bound.*** An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free.**

Ex. 18(44): In the statement $\exists x(x + y = 1)$, *the variable x is bound by the existential quantification $\exists x$, but* the variable *y is free because it is not bound by a quantifier and no value is assigned to this* variable.

Definition 3(45): Statements involving predicates and quantifiers are *logically equivalent if and only if they* have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ *to indicate that two statements S and T involving predicates and* quantifiers are logically equivalent.

Negating Quantified Expressions: Consider the negation of the statement

"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely, $\forall x P(x)$, where *P(x) is the statement "x has taken a course in calculus" and the domain consists of the* students in your class. The negation of this statement is "It is not the case that every student in your class has taken a course in calculus." This is equivalent to "There is a student in your class who has not taken a course in calculus." And this is simply the existential quantification of the negation of the original propositional function, namely, $\exists x \neg P(x)$.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Suppose we wish to negate an existential quantification. For instance, consider the proposition "There is a student in this class who has taken a course in calculus." This is the existential quantification $\exists x Q(x)$, where *Q(x) is the statement "x has taken a course in calculus." The negation of this statement* is the proposition "It is not the case that there is a student in this class who has taken a course in calculus." This is equivalent to "Every student in this class has not taken calculus," which is just the universal quantification of the negation of the original propositional function, or, phrased in the language of quantifiers, $\forall x \neg Q(x)$.

$$\neg \exists x Q(x) \equiv \forall x \neg Q(x)$$

Quantifiers in terms of De Morgan's Laws
Negation of $\forall x\, P(x)$ *is* "$\neg\exists x$ P(x) " which is equal to "$\forall x \neg$ P(x) ": True if, for every x, P(x) is false; False if, there is an x for which P(x) is true.
Negation of $\exists x$ P(x) *is* "$\neg\forall x$ P(x) " which is equal to "$\exists x \neg$ P(x) ": True if, there is an x, for which P(x) is false; False if, P(x) is true for every x.

Remark: When the domain of a predicate P(x) consists of n elements, where n is a positive integer greater than one, the rules for negating quantified statements are exactly the same as De Morgan's laws discussed in Section 1.3. This is why these rules are called De Morgan's laws for quantifiers. When the domain has *n elements x1, x2, . . . , xn, it follows that* $\neg\forall x P(x)$ is the same as $\neg(P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n))$, *which is equivalent to* $\neg P(x1) \vee \neg P(x2) \vee \cdots \vee \neg P(xn)$ *by De Morgan's laws, and this is the same as* $\exists x \neg P(x)$. *Similarly,* $\neg\exists x P(x)$ is the same as $\neg(P(x1) \vee P(x2) \vee \cdots \vee P(xn))$, *which by De Morgan's laws is equivalent to* $\neg P(x1) \wedge \neg P(x2) \wedge \cdots \wedge \neg P(xn)$, *and this is the same as* $\forall x \neg P(x)$.

Ex. 20(47): What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?
Solution:

Ex. 21(47): What are the negations of the statements $\forall x(x^2 > x)$ *and* $\exists x(x^2 = 2)$*?*
Solution:

Ex. 22(48): Show that $\neg\forall x(P(x) \rightarrow Q(x))$ *and* $\exists x(P(x) \wedge \neg Q(x))$ *are logically equivalent.*
Solution:

Ex. 23(48): Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
Solution:

Ex. 24(49): Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.
Solution:

Ex. 27(51): Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly colored."
"No large birds live on honey."
"Birds that do not live on honey are dull in color."
"Hummingbirds are small."

Let *P(x), Q(x), R(x), and S(x) be the statements "x is a hummingbird," "x is large," "x lives on* honey," and "*x is richly colored," respectively. Assuming that the domain consists of all birds,* express the statements in the argument using quantifiers and *P(x), Q(x), R(x), and S(x).*
Solution:

Nested Quantifiers : Nested quantifiers commonly occur in mathematics and computer science. This quantifiers si represented such as structure which one quantifier is within the scope of another.

$$\forall x \exists y(x + y = 0)$$

Ex. 1(57): Assume that the domain for the variables *x and y consists of all real numbers.*
Solution: $\forall x \forall y(x + y = y + x)$ *or* $\forall x \exists y(x + y = 0)$*(it says* that for every real number *x there is a real number y such that* $x + y = 0$*.)*

Ex. 2(58): Translate into English the statement of $\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$.
Solution:

Ex. 4(58): Let Q(x, y) denote "x + y = 0." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?
Solution:

Quantifications of two variables
"∀x ∀y P(x,y)" or "∀y ∀x P(x,y)" is True when P(x,y) is True for every pair (x-y); is False when there is a pair (x-y) for which P(x,y) is False.
"∀x ∃y P(x,y)" is True for every x there is a y for which P(x,y) is True; is False when there is an x such that P(x,y) is false for every y.
"∃x ∀y P(x,y)" is True when there is an x for which P(x,y) is True for every y; is False for every x there is a y for which P(x,y) is false.
"∃x ∃y P(x,y)" or "∃y ∃x P(x,y)" is True when there is a pair (x-y) for which P(x,y) is True; is False when P(x,y) is false for every pair (x-y).

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Ex. 6(60): Translate the statement "The sum of two positive integers is always positive" into a logical expression.
Solution:

Ex. 7(60): Translate the statement "Every real number except zero has a multiplicative inverse." (A multiplicative inverse of a real number x is a real number y such that xy = 1.)
Solution:

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Ex. 9(61): Translate the statement $\forall x(C(x) \lor \exists y(C(y) \land F(x, y)))$ *into* English where C(x) is "x has a computer," F(x, y) is "x and y are friends," and the domain for both x and y consists of all students in your school.
Solution:

Ex. 10(61): Translate the statement $\exists x \forall y \forall z((F(x, y) \land F(x, z) \land (y \neq z)) \rightarrow \neg F(y,z))$ into English, where *F(a,b) means a and b are friends and the domain for x, y, and z consists of* all students in your school.
Solution:

Translating English Sentences into Logical Expressions

Ex. 11(62): *Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.*
Solution:

Ex. 12(62): *Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.*
Solution:

Negating Nested Quantifiers

Ex. 14(63): Express the negation of the statement $\forall x \exists y(xy = 1)$ *so that no negation precedes a quantifier.*
Solution:

Ex. 15(63): Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every "airline in the world."
Solution:

Rules of Inference

• **Proofs** in mathematics are valid arguments that establish the truth of mathematical statements.

• **Argument** means a sequence of statements that end with a conclusion.

• **Valid** means that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or premises, of the argument.

•An argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.

•To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. *Rules of inference are our basic tools for establishing the truth of statements.*

•The mathematical proof of logical statements must necessitates the following steps:

- ➢The arguments which involve only compound proposition should be considered
- ➢The definition of meaning for an argument involving compound propositions to be valid.
- ➢introduce a collection of rules of inference in propositional logic.
- ➢After studying rules of inference in propositional logic, rules of inference for quantified statements should be introduced(existential and universal quantifiers play an important role).
- ➢Finally, it should be determined that how rules of inference for propositions and for quantified statements can be combined systematically.

Valid Arguments in Propositional Logic: Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

"If you have a current password, then you can log onto the network."
"You have a current password."
Therefore,
"You can log onto the network."

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion "You can log onto the network" must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

If we specify the ""p"" to represent "You have a current password" and ""*q*"" *to represent "You can log onto the network."* Then, the argument has the form

$$p \to q$$
$$p$$
$$\therefore q$$

where $\therefore$ is the symbol that denotes "therefore."

We know that when *p and q are propositional variables, the statement* $((p \to q) \wedge p) \to q$ is a tautology (see Exercise 10(c) in Section 1.3). In particular, when both $p \to q$ *and p are* true, we know that *q must also be true. We say this form of argument is* ***valid because whenever*** all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true.

Now suppose that both "If you have a current password, then you can log onto the network" and "You have a current password" are true statements. When we replace *p by "You have a current password" and q by "You can log onto the network," it* necessarily follows that the conclusion "You can log onto the network" is true. This argument is **valid because its form is valid. Note that whenever we replace *p and q* by propositions where** $p \to q$ *and p are both true, then q must also be true.*

What happens when we replace *p and q in this argument form by propositions where not* both *p and* $p \to q$ *are true? For example, suppose that p represents "You have access to the* network" and *q represents "You can change your grade" and that p is true, but* $p \to q$ *is false.* The argument we obtain by substituting these values of *p and q into the argument form is*

"If you have access to the network, then you can change your grade."
"You have access to the network."
$\therefore$ "You can change your grade."

The argument we obtained is a valid argument, but because one of the premises, namely the first premise, is false, we cannot conclude that the conclusion is true.

<u>Definition 1(70):</u>An *argument in propositional logic is a sequence of propositions. All but the final Proposition* in the argument are called *premises and the final proposition is called the conclusion. An* argument is *valid if the truth of all its premises implies that the conclusion is true.*

An *argument form in propositional logic is a sequence of compound propositions involving* propositional variables. An argument form is *valid no matter which particular propositions* are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

- Note that, from the definition of a valid argument form we see that the argument form with premises *p_1, p_2, . . . , p_n and conclusion q is valid, when $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology.*

Rules of Inference for Propositional Logic

•We can always use a truth table to show that an argument form is valid. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires 1024 different rows.

•Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference.**

•These rules of inference can be used as building blocks to construct more complicated valid argument forms.

•Most important rules of inference in propositional logic will be introduced.

•The tautology *$(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called* ***modus ponens,*** or the **law of detachment. (Modus ponens is Latin for *mode that affirms.) This tautology*** leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol ∴ denotes "therefore"):

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

Ex. 1: Suppose that the conditional statement "If it snows today, then we will go skiing" and it is hypothesis, "It is snowing today," are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, "We will go skiing," is true.

Ex. 2: Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument .

"If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$."

Solution: Let p be the proposition "$\sqrt{2} > \frac{3}{2}$" and q the proposition "$2 > (\frac{3}{2})^2$." The premises of the argument are $p \rightarrow q$ and p, and q is its conclusion. This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premises, $\sqrt{2} > \frac{3}{2}$, is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$. ◀

Ex. 3: State which rule of inference is the basis of the following argument: "It is below freezing now. Therefore, it is either below freezing or raining now."

Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{l} \underline{\quad p \wedge q} \\ \therefore p \end{array}$$

This argument uses the simplification rule. ◀

✓ **There are many useful rules of inference for propositional logic. Perhaps the most widely used of these are listed as follows.**

Rule of Inference	***Tautology***	***Name***
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Ex. 4: State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."

Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{l} \quad p \land q \\ \hline \therefore p \end{array}$$

This argument uses the simplification rule. ◀

Ex. 5 : State which rule of inference is used in the following argument:

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{l} \quad p \to q \\ \quad q \to r \\ \hline \therefore p \to r \end{array}$$

Hence, this argument is a hypothetical syllogism. ◀

Rules of Inference for Quantified Statements

Universal instantiation is the rule of inference used to conclude that $P(c)$ *is true, where c* is a particular member of the domain, given the premise $\forall xP(x)$. *Universal instantiation is used* when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

Universal generalization is the rule of inference that states that $\forall xP(x)$ *is true, given the* premise that $P(c)$ *is true for all elements c in the domain. Universal generalization is used when* we show that $\forall xP(x)$ *is true by taking an arbitrary element c from the domain and showing that* $P(c)$ *is true. The element c that we select must be an arbitrary, and not a specific, element of* the domain. That is, when we assert from $\forall xP(x)$ *the existence of an element c in the domain,* we have no control over *c and cannot make any other assumptions about c other than it comes* from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element *c when universal generalization is used is all too common in incorrect* reasoning.

Existential instantiation is the rule that allows us to conclude that there is an element *c in* the domain for which $P(c)$ *is true if we know that* $\exists xP(x)$ *is true.We cannot select an arbitrary* value of *c here, but rather it must be a c for which* $P(c)$ *is true. Usually we have no knowledge* of what *c is, only that it exists. Because it exists, we may give it a name (c) and continue our* argument.

Existential generalization is the rule of inference that is used to conclude that $\exists xP(x)$ *is* true when a particular element *c with* $P(c)$ *true is known. That is, if we know one element c in* the domain for which $P(c)$ *is true, then we know that* $\exists xP(x)$ *is true.*

Following table summarizes these rules:

Rule of Inference	***Name***
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

Ex. 12: Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

◀

Ex. 13: Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

◀

Combining Rules of Inference for Propositions and Quantified Statements:

- Rules of inference both for propositions and for quantified statements have been developed.
- Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called universal modus ponens. This rule tells us that if $\forall x(P(x) \rightarrow Q(x))$ *is true, and if* $P(a)$ *is* true for a particular element *a in the domain of the universal quantifier, then* $Q(a)$ *must also* be true. To see this, note that by universal instantiation, $P(a) \rightarrow Q(a)$ *is true. Then, by modus* ponens, $Q(a)$ *must also be true.We can describe universal modus ponens as follows:*

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ \underline{P(a)(a \text{ is particular element in the domain})} \\ \therefore Q(a) \end{array}$$

Ex. 14: Assume that "For all positive integers *n, if n is greater than 4, then* n^2 *is less than 2n" is true.* Use universal modus ponens to show that $100^2 < 2^{100}$.

Solution: Let $P(n)$ denote "$n > 4$" and $Q(n)$ denote "$n^2 < 2^n$." The statement "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be represented by $\forall n(P(n) \rightarrow Q(n))$, where the domain consists of all positive integers. We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true. Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(100)$ is true, namely that $100^2 < 2^{100}$. ◀