Discrete Mathematics, KOM1062
Lecture #4

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Lecture Book: **"Discrete Mathematics, Seventh Edt., Kenneth H. Rosen, 2007, McGraw Books, Discrete Mathematics and Applications, Susanna S. Epp, Brooks, 4th Edt., 2011".**

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Advanced Counting Techniques

Recurrence Relations: *An argument can* be given that shows the sequence $\{a_n\}$ *satisfies the recurrence relation* $a_{n+1} = a_n + a_{n-1}$ and the initial conditions $a_1 = 2$ *and* $a_2 = 3$.

This recurrence relation and the initial conditions determine the sequence $\{a_n\}$. *Moreover, an explicit formula can be found for* a_n *from the equation* relating the terms of the sequence.

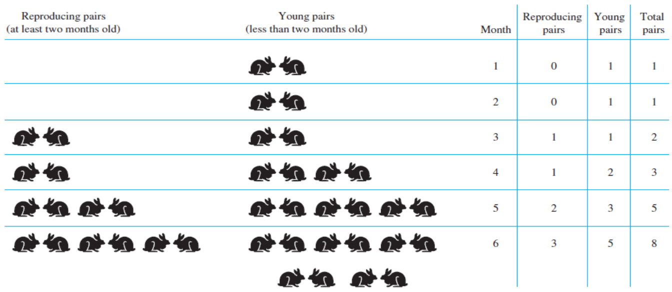
We will show that such relations can be used to study and to solve counting problems. For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in *n hour?*

To solve this problem, let *an be the number of bacteria at the end of n hours. Because the number of bacteria doubles* every hour, the relationship $a_n = 2a_{n-1}$ *holds whenever n is a positive integer. This recurrence* relation, together with the initial condition $a_0 = 5$, *uniquely determines* a_n *for all nonnegative* integers *n.*

Advanced Counting Techniques

Ex. 1: (***Rabbits and the Fibonacci Numbers)*** *Consider this problem, which was originally posed by Leonardo Pisano:*
A young pair of rabbits (one of each sex) is placed on an island.
A pair of rabbits does not breed until they are 2 months old.
After they are 2 months old, each pair of rabbits produces another pair each month, as shown in the Figure below.

Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.



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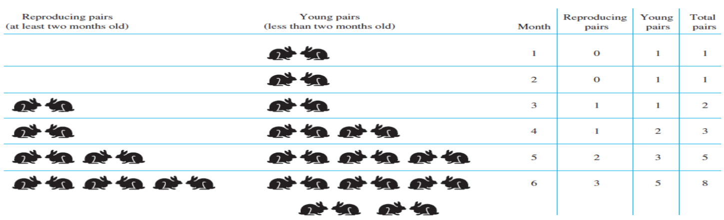
Solution: Denote by f_n *the number of pairs of rabbits after n months. We will show that* f_n*, n = 1, 2, 3, . . . , are the terms of the Fibonacci sequence.*
The rabbit population can be modeled using a recurrence relation. At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$. *Because this pair does not* breed during the second month, $f_2 = 1$ *also.*
To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, *and the number of newborn pairs, which* equals f_{n-2}, *because each newborn pair comes from a pair at least 2 months old.* Consequently, the sequence $\{f_n\}$ *satisfies the recurrence relation*

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 3$ *together with the initial conditions* $f_1 = 1$ *and* $f_2 = 1$. *Because this recurrence relation* and the initial conditions uniquely determine this sequence, the number of pairs of rabbits on the island after *n months is given by the nth Fibonacci number.*



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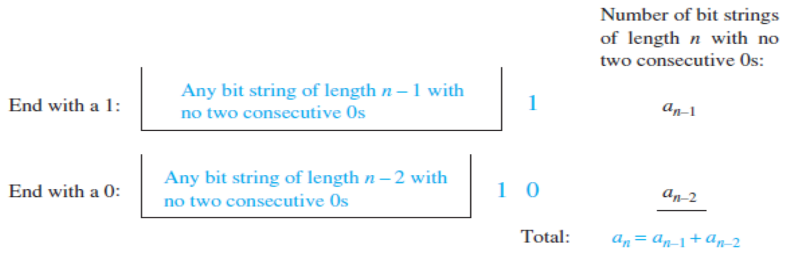
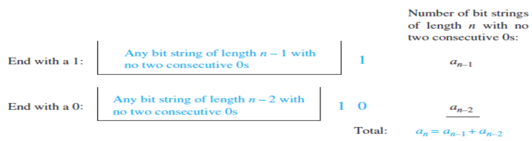
Ex. 2: Find a recurrence relation and give initial conditions for the number of bit strings of length *n* that *do not have two consecutive 0s*. How many such bit strings are there of length five?



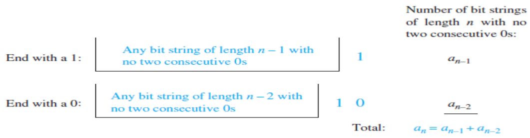


Solution: Let a_n *denote the number of bit strings of length n that do not have two consecutive 0s.*

To obtain a recurrence relation for $\{a_n\}$, *note that by the sum rule, the number of bit strings of* length *n that do not have two consecutive 0s equals the number of such bit strings ending with* a “0” plus the number of such bit strings ending with a “1”. We will assume that $n \geq 3$, *so that the* bit string has at least three bits.

The bit strings of length *n ending with 1 that do not have two consecutive 0s are precisely the* bit strings of length *n − 1 with no two consecutive 0s with a “1” added at the end.*

Consequently, there are a_{n-1} *such bit strings.* Bit strings of length *n ending with a “0” that do not have two consecutive 0s must have 1* as their *(n − 1)st bit; otherwise they would end with a pair of 0s. It follows that the bit strings* of length *n ending with a 0 that have no two consecutive 0s are precisely the bit strings of* length *n − 2 with no two consecutive 0s with “10” added at the end. Consequently, there are* a_{n-2} such bit strings.



Solution: (continued)

We conclude, as illustrated in the Figure, that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.
The initial conditions are $a_1 = 2$, *because both bit strings of length one, 0 and 1 do not have* consecutive 0s, and $a_2 = 3$, *because the valid bit strings of length two are 01, 10, and 11. To obtain* a_5, *we use the recurrence relation three times to find that*

$a_3 = a_2 + a_1 = 3 + 2 = 5,$
$a_4 = a_3 + a_2 = 5 + 3 = 8,$
$a_5 = a_4 + a_3 = 8 + 5 = 13.$

Note that: $a_3 = a_2 + a_1 = \{(01), (10), (11)\}+\{(0),(1)\}=5$
$a_4 = a_3 + a_2 = \{(010), (011), (101), (110), (111)\}+\{(0),(1)\}=8$
$a_5 = a_4 + a_3 = \{(010), (011), (101), (110), (111)\}+\{(01), (10), (11)\}=13$

Ex. 3: A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits.
For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n *be the number of valid n-digit codewords.*

Find a recurrence relation for *an.*

Solution:

Note that $a_1 = 9$ *because there are 10 one-digit strings, and only one, namely, the* string 0, is not valid.

A recurrence relation can be derived for this sequence by considering how a valid *n-digit string can be obtained from strings of n − 1 digits.*

There are two ways to form a valid string with *n digits from a string with one fewer digit.*

Solution: (Continued…)

First, a valid string of *n digits can be obtained by appending a valid string of n − 1 digits* with a digit other than 0. This appending can be done in nine ways. Hence, a valid string with *n digits can be formed in this manner in* $9a_{n-1}$ *ways.*

Second, a valid string of *n digits can be obtained by appending a 0 to a string of length n − 1 that is not valid. (This produces a string with an even number of 0 digits because the* invalid string of length *n − 1 has an odd number of 0 digits.) The number of ways that this can* be done equals the number of invalid *(n − 1)-digit strings. Because there are* 10^{n-1} *strings of* length *n − 1, and* a_{n-1} *are valid, there are* $10^{n-1} - a_{n-1}$ *valid n-digit strings obtained by* appending an invalid string of length *n − 1 with a 0.* Because all valid strings of length *n are produced in one of these two ways, it follows that* there are

$$a_n = 9_{an-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$$

valid strings of length *n.*

Solving Linear Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k},$$

where $c_1, c_2, \ldots, c_k$ *are real numbers, and* $c_k \neq 0$.

The recurrence relation in the definition is linear because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of *n. The recurrence relation is* homogeneous because no terms occur that are not multiples of the a_j*s. The coefficients of the* terms of the sequence are all constants, rather than functions that depend on *n. The degree* is *k because* a_n *is expressed in terms of the previous k terms of the sequence.*

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the *k initial conditions:*

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$$

Ex. 1:
The recurrence relation $P_n = (1.11)P_{n-1}$ *is a linear homogeneous recurrence relation of degree* one.

The recurrence relation $f_n = f_{n-1} + f_{n-2}$ *is a linear homogeneous recurrence relation of* degree two.

The recurrence relation $a_n = a_{n-5}$ *is a linear homogeneous recurrence relation of* degree five.

Ex. 2:
The recurrence relation $an = a_{n-1} + a_{n-2}^2$ *is not linear.*

The recurrence relation Hn = 2*Hn−1 +1 is not homogeneous.*

The recurrence relation Bn = nBn−1 does not have constant coefficients.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, *where r is a constant. Note that* $a_n = r^n$ *is a solution of the recurrence* relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ *if and only if*

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k} .$$

When both sides of this equation are divided by r^{n-k} *and the right-hand side is subtracted from* the left, we obtain the equation

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0.$$

Consequently, the sequence {*an*} *with* $a_n = r^n$ *is a solution if and only if r is a solution of this* last equation. We call this the characteristic equation of the recurrence relation. The solutions of this equation are called the characteristic roots of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

We will first develop results that deal with linear homogeneous recurrence relations with constant coefficients of degree two. Then corresponding general results when the degree may be greater than two will be stated.

Theorem 1: Let c_1 *and* c_2 *be real numbers. Suppose that* $r^2 - c_1r - c_2 = 0$ *has two distinct roots* r_1 and r_2. *Then the sequence {an} is a solution of the recurrence relation* $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1 r_1^{n+} \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, *where* α_1 *and* α_2 *are constants.*

Proof: We must do two things to prove the theorem. First, it must be shown that if *r*1
and *r*2
are the roots of the characteristic equation, and *α*1 and *α*2 are constants, then the
sequence {*an*}
with *an* = *α*1*rn*
1
+ *α*2*rn*
2 is a solution of the recurrence relation. Second, it must be shown that
if the sequence {*an*} is a solution, then *an* = *α*1*rn*
1
+ *α*2*rn*
2 for some constants α1 and α2

This shows that the sequence {*an*} with *an* = *α*1*rn*
1
+ *α*2*rn*
2 is a solution of the recurrence relation.

To show that every solution {*an*} of the recurrence relation *an* = *c*1*an*−1 + *c*2*an*−2
has *an* = *α*1*rn*
1
+ *α*2*rn*
2 for *n* = 0, 1, 2, . . . , for some constants *α*1 and *α*2, suppose that {*an*} is a
solution of the recurrence relation, and the initial conditions *a*0 = *C*0 and *a*1 = *C*1 hold. It will
be shown that there are constants *α*1 and *α*2 such that the sequence {*an*} with *an* = *α*1*rn*
1
+ *α*2*rn*
2
satisfies these same initial conditions. This requires that
*a*0 = *C*0 = *α*1 + *α*2,
*a*1 = *C*1 = *α*1*r*1 + *α*2*r*2.
We can solve these two equations for *α*1 and *α*2. From the first equation it follows that
*α*2 = *C*0 − *α*1. Inserting this expression into the second equation gives
*C*1 = *α*1*r*1 + (*C*0 − *α*1)*r*2.
Hence,
*C*1 = *α*1(*r*1 − *r*2) + *C*0*r*2.
This shows that
*α*1 = *C*1 − *C*0*r*2

$a_n = c_1a_{n-1} + c_2a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$. Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for

all nonnegative integers n. We have completed the proof by showing that a solution of the linear
homogeneous recurrence relation with constant coefficients of degree two must be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants.
The characteristic roots of a linear homogeneous recurrence relation with constant coefficients
may be complex numbers. Theorem 1 (and also subsequent theorems in this section) still applies in this case. Recurrence relations with complex characteristic roots will not be discussed
in the text. Readers familiar with complex numbers may wish to solve Exercises 38 and 39.
Examples 3 and 4 show how to use Theorem 1 to solve recurrence relations.

Note that, the characteristic roots of a linear homogeneous recurrence relation with constant coefficients may be complex numbers. Theorem 1 (and also subsequent theorems in this section) still applies in this case.3

Ex. 3: What is the solution of the recurrence relation
$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ *and* $a_1 = 7$?

Solution: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$. Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{an\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n$$

with coefficients α_1 , α_2. From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2,$$
$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$$

Then we have $\alpha_1 = 3$, $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{an\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n$$

Ex. 4: Find an explicit formula for the Fibonacci numbers.

Solution: Recall that the sequence of Fibonacci numbers satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

and also satisfies the initial conditions

$$f_0 = 0 \text{ and } f_1 = 1$$

The roots of the characteristic equation:

$$r^2 - r - 1 = 0$$

$$r_1 = (1 + \sqrt{5})/2 \text{ and } r_2 = (1 - \sqrt{5})/2.$$

Therefore, from Theorem it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

with coefficients α_1 , α_2. *With the initial conditions, we have*

$$f_0 = \alpha_1 + \alpha_2 = 0,$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Solution: (cont.)…

The solution to these simultaneous equations for α_1 , α_2 , we obtain

$$\alpha_1 = 1/\sqrt{5}, \qquad \alpha_2 = -1/\sqrt{5}.$$

Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Theorem 2: (characteristic root of multiplicity two**)**
Let c_1 *and* c_2 *be real numbers with* $c_2 \neq 0$. *Suppose that* $r^2 - c_1 r - c_2 = 0$ *has one root* r_0. *Then the sequence {an} is a solution of the recurrence relation* $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \ldots$, *where* α_1 *and* α_2 *are constants.*

Ex. 5: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ *and* $a_1 = 6$?
Solution:
$r^2 - 6r + 9 = 0 \qquad r = 3$
Hence, the solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 *and* α_2. Using the initial conditions, it follows that

$$a_0 = 1 = \alpha_1,$$
$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving these two equations shows that $\alpha_1 = 1$ *and* $\alpha_2 = 1$. Consequently, the solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n$$

Theorem 3: Let $c_1, c_2, \ldots, c_k$ *be real numbers .Suppose that the characteristic equation* $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ *has k distinct roots* $r_1, r_2, \ldots, r_k$. *Then a sequence {an} is a solution of the recurrence relation* $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 n r_2^n + \ldots + \alpha_k r_k^n$ for $n = 0, 1, 2, \ldots$, *where* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are constants.*

Ex. 6: Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0 = 2$ *and*
$a_1 = 5$ *and* $a_2 = 15$?
Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are

$$r = 1, \quad r = 2, \quad \text{and} \quad r = 3.$$

Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions:

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$$
$$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$$
$$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9.$$

Solution: (Cont.) When these three simultaneous equations are solved for α_1 , α_2, and α_3 we find those coefficients are 1, -1, and 2, respectively.

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence {*an*} with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

We now state the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have multiple roots.

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers .Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has t distinct roots r_1 , $r_2, \ldots, r_t$ with multiplicities m_1 , $m_2, \ldots, m_t$,respectively, so that m_i >= 1 for i= 1,2,3,…,t

and $m_1+m_2+ \ldots + m_t = k$. Then a sequence $\{a_n\}$ i a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

İf and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \ldots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for *n = 0, 1, 2, . . . , where* $\alpha_{i,j}$ *are constants for* $1 \le i \le t$ *and* $0 \le j \le m_{i-1}$.

Ex. 7: Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

Solution: The general form of the solution is

$$(\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)2^n + (\alpha_{2,0} + \alpha_{2,1}n)5^n + \alpha_{3,0}9^n.$$

Ex. 8(520): Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

There is a single root $r = -1$ of multiplicity three of the characteristic equation.

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

$$a_0 = 1 = \alpha_{1,0},$$
$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2},$$
$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}.$$

$$\alpha_{1,0} = 1,\ \alpha_{1,1} = 3,\ \text{and}\ \alpha_{1,2} = -2$$

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence {an} with

$$a_n = (1 + 3n - 2n^2)(-1)^n$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The recurrence relation $a_n = 3_{an-1} + 2n$ *is an example of a linear nonhomogeneous* recurrence relation with constant coefficients, that is, a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdot\cdot\cdot + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots, c_k$ *are real numbers and* $F(n)$ *is a function not identically zero depending* only on n. *The recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdot\cdot\cdot + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation. It plays an important role in the solution of the nonhomogeneous recurrence relation.

Ex. 9: Each of the recurrence relations

$$a_n = a_{n-1} + 2^n,$$
$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1,$$
$$a_n = 3a_{n-1} + n3^n, \text{ and}$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

is a linear nonhomogeneous recurrence relation with constant coefficients. The associated linear homogeneous recurrence relations are

$$a_n = a_{n-1},$$
$$a_n = a_{n-1} + a_{n-2},$$
$$a_n = 3a_{n-1}, \text{ and}$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3},$$

respectively.

The key fact about linear nonhomogeneous recurrence relations with constant coefficients is that every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation, as Theorem 5 shows.

Theorem 5: If $\{a_n^{(p)}\}$ *is a particular solution of the nonhomogeneous linear recurrence relation with*
constant coefficients

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, *where* $\{a_n^{(h)}\}$ *is a solution of the associated* homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}.$$

By Theorem 5, we see that the key to solving nonhomogeneous recurrence relations with constant coefficients is finding a particular solution. Then every solution is a sum of this solution and a solution of the associated homogeneous recurrence relation. Although there is no general method for finding such a solution that works for every function *F(n), there are techniques that* work for certain types of functions *F(n), such as polynomials and powers of constants.*

Ex. 10(522): Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. *What is the solution with* $a_1 = 3$?
Solution: We need to solve its associated linear homogeneous equation and to find a particular solution for the given nonhomogeneous equation.

The associated linear homogeneous equation and the related solutions are:

$$a_n = 3a_{n-1}$$

$$a_n^{(h)} = \alpha 3^n$$

For particular solution,

$$F(n) = 2n$$

then

$$p_n = cn + d,$$

where c and d are constants.

$$a_n = 3a_{n-1} + 2n \qquad cn + d = 3(c(n-1) + d) + 2n$$

We get c=-1 and d=-1.5: $a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n$
For n=1 and $a_1 = 3$, *we obtain:*

$$a_n = -n - 3/2 + (11/6)3^n$$

Ex. 11(522): Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7n.$$

Solution:

This is a linear nonhomogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$$a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n \qquad F(n) = 7^n$$

A reasonable trial solution: $a_n^{(p)} = C \cdot 7^n$

Substituting the terms of this sequence into the recurrence relation implies that

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$$

We get C=49/20. Hence, the particular solution :

$$a_n^{(p)} = (49/20)7^n$$

General solution: $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n$

In the last two, we made an educated guess that there are solutions of a particular form. In both cases we were able to find particular solutions. Thiswas not an accident. Whenever *F(n) is the product of a polynomial in n and the nth power of a constant, we know exactly what* form a particular solution has, as stated in Theorem 6.

Theorem 6: Suppose that $\{a_n\}$ *satisfies the linear nonhomogeneous recurrence relation*

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n),$$

where c_1 , c_2, …, c_k are real numbers, and

$$F(n)= (b_tn^t+ b_{t-1}n^{t-1}+\ldots+ b_1n+b_0)s^n$$

where $b_0,\ldots, b_t$ and *s are real numbers.When s is not a root of the characteristic equation* of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_tn^t\ p_{t-1}n^{t-1}+\ldots+ p_1n+p_0)s^n.$$

When *s is a root of this characteristic equation and its multiplicity is m, there is a particular* solution of the form

$$n^m(p_tn^t\ p_{t-1}n^{t-1}+\ldots+ p_1n+p_0)s^n.$$

Note that in the case when *s is a root of multiplicity m of the characteristic equation of* the associated linear homogeneous recurrence relation, the factor *nm ensures that the proposed* particular solution will not already be a solution of the associated linear homogeneous recurrence relation.

Ex. 12(523): What form does a particular solution of the linear nonhomogeneous recurrence relation *$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^22^n$, and $F(n) =(n^2 + 1)3^n$?*

Solution:
The associated linear homogeneous recurrence relation is *an* = 6*an*−1 − 9*an*−2. Its characteristic equation, *r*2 − 6*r* + 9 = *(r* − 3*)*2 = 0, has a single root, 3, of multiplicity two. To apply Theorem 6, with *F(n)* of the form *P(n)sn*, where *P(n)* is a polynomial and *s* is a constant, we need to ask whether *s* is a root of this characteristic equation.
Because *s* = 3 is a root with multiplicity*m* = 2but *s* = 2 is not a root, Theorem 6 tells us that a particular solution has the form *p*0*n*23*n* if *F(n)* = 3*n*, the form *n*2*(p*1*n* + *p*0*)*3*n* if *F(n)* = *n*3*n*, the form *(p*2*n*2 + *p*1*n* + *p*0*)*2*n* if *F(n)* = *n*22*n*, and the form *n*2*(p*2*n*2 + *p*1*n* + *p*0*)*3*n* if *F(n)* = *(n*2 + 1*)*3*n*.
▲
Care must be taken when *s* = 1 when solving recurrence relations of the type covered by Theorem 6. In particular, to apply this theorem with *F(n)* = *btnt* + *bt*−1*nt*−1 +· · ·+*b*1*n* + *b*0, the parameter *s* takes the value *s* = 1 (even though the term 1*n* does not explicitly appear). By the theorem, the form of the solution then depends on whether 1 is a root of the characteristic equation of the associated linear homogeneous recurrence relation. This is illustrated in Example 13, which shows how Theorem 6 can be used to find a formula for the sum of the first *n* positive integers.

Ex. 13(523): Let a_n *be the sum of the first n positive integers, so that*

$$a_n = \sum_{k=1}^{n} k.$$

Note that an satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n$. (To obtain an, the sum of the first n positive integers, from an−1, the sum of the first n − 1 positive integers, we add n.) Note that the initial condition is $a_1 = 1$. The associated linear homogeneous recurrence relation for an is $a_n = a_{n-1}$. The solutions of this homogeneous recurrence relation are given by $a_n^{(h)} = c(1)^n = c$, where *c is a constant. To find all solutions of $a_n = a_{n-1} + n$, we need find only a single particular* solution. By Theorem 6, because $F(n) = n = n \cdot (1)^n$ *and s = 1 is a root of degree one of* the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form $n(p_1n + p_0) = p_1n^2 + p_0n$.

Inserting this into the recurrence relation gives $p_1n^2 + p_0n = p_1(n-1)^2 + p_0(n-1) + n$. *Simplifying, we see that* $n(2p_1 - 1) + (p_0 - p_1) = 0$, *which means* that $2p_1 - 1 = 0$ *and* $p_0 - p_1 = 0$, *so* $p_0 = p_1 = 1/2$. *Hence,*

$$a_n^{(p)} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

is a particular solution. Hence, all solutions of the original recurrence relation $a_n = a_{n-1} + n$ *are* given by $a_n = a_n^{(h)} + a_n^{(p)} = c + n(n+1)/2$. *Because $a_1 = 1$, we have $1 = a_1 = c + 1 \cdot 2/2 = c + 1$, so $c = 0$. It follows that an = n(n + 1)/2.*