System Theory, KOM5108
Lecture #4

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Lecture Book: **"Linear System Theory, W.J. Rugh''**

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State Equation Representation

The basic representation for linear systems is the linear state equation in the following standard from

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

where $A(t)_{n\times n}, B(t)_{n\times m}, C(t)_{p\times n}$ and $D(t)_{p\times m}$ are continuous, real-valued functions defined for all $t \in (-\infty, \infty)$. If those coefficient matrices are constant, then it defines a time invariant system. Therefore, the linear equations are called as time varying if any entry of any coefficient matrix varies with time.

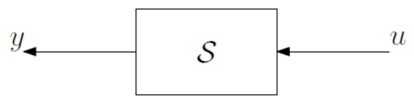
For practical problems, there is a fixed initial time t_0, and the properties of the solution $x(t)$ of a linear state equation for given initial state $x(t_0) = x_0$ and input signal $u(t)$ specified for $t \in (t_0, \infty)$ are of interest for $t \geq t_0$.

However from a mathematical viewpoint there are occasions when solutions backward in time are of interest, and this is the reason that the interval of the definition of the input signal and coefficient matrices in the state equation is $(-\infty, \infty)$.

Hence, the solution of the $x(t)$ for $t < t_0$, as well as $t \geq t_0$, mathematically valid.

Moreover, if the state equation is defined and of interest only in a smaller interval as $t \in [0, \infty)$ the domain of definition of the coefficient matrices can be extended to $(-\infty, \infty)$. $(\text{e.g.}, A(t) = A(0)$ for $t < 0)$.

A system is a mapping from an input linear vector space U to an output Y and usually it is customary to use block diagrams to descibe this relationship.



Definition: A system($\mathcal{S}$) is said to be

(a) Linear if:

$$\mathcal{S}(\alpha u) = \alpha S \text{ for all } u \in \mathcal{U} \text{ and } \alpha \in \mathbb{F}$$

$$\mathcal{S}(u_1 + u_2) = \mathcal{S}(u_1) + \mathcal{S}(u_2) \text{ for all } u_1, u_2 \in \mathcal{U}.$$

(b) Nonlinear if it is not linear.

(c) Time-invariant if

$$\mathcal{S}(u(t)) =: y(t) \quad \text{implies} \quad \mathcal{S}(u(t+T)) = y(t+T) \quad \text{for all } u \in \mathcal{U}, \text{ for all } T.$$

(d) Time-varying if it is not time-invariant

(e) Causal if, the output depends on past and current inputs but not future inputs.

Ex. : Time-varying versions of the basic linear circuit elements can be devised in simple ways. A time-varying resistor exhibits the voltage/current charactersitic

$$v(t) = r(t)i(t)$$

where $r(t)$ is a fixed time function. For ex., if $r(t)$ is sinusoid, then this is the basis for some modulation schemes in commnication systems.

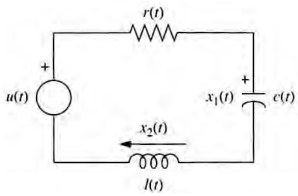
A time-varying capacitor exhibits a time-varying charge/voltage characteristic, $\mathrm{q}(t) = c(t)v(t)$. Here, $c(t)$ is fixed time function describing, for ex., the variation in plate spacing of a parallel-plate capacitor. Since current is the instantaneous rate of change of charge, the voltage/current relationship for a time-varying capacitor has the form

$$i(t) = c(t)\frac{dv(t)}{dt} + \frac{dc(t)}{dt}v(t)$$

In a similar way, a time-varying inductor exhibits a time-varying flux/current characteristic, and this leads to the voltage/current relation.

$$v(t) = l(t)\frac{di(t)}{dt} + \frac{dl(t)}{dt}i(t)$$

Ex. : Consider the series circuit shown in the Figure.



Suppose that the output signal $y(t)$ is the voltage across the resistor and choose the state variables as the voltage $x_1(t)$ across the capacitor and the current $x_2(t)$ through the inductor(current through the entire series circuit). Then the Kichhoff's voltage law for this circuit gives the following equations and linear system description:

$$\dot{x_2}(t) = \frac{-1}{l(t)}\left[r(t) + \dot{l}(t)\right]x_2(t) + \frac{1}{l(t)}u(t)$$

$$\dot{x_1}(t) = \frac{-\dot{c}(t)}{c(t)}x_1(t) + \frac{1}{c(t)}x_2(t)$$

$$y(t) = r(t)x_2(t)$$

$$A(t) = \begin{bmatrix} \frac{-\dot{c}(t)}{c(t)} & \frac{1}{c(t)} \\ \frac{-1}{l(t)} & \frac{-r(t)-\dot{l}(t)}{l(t)} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ \frac{1}{l(t)} \end{bmatrix}, \quad C(t) = \begin{bmatrix} 0 & r(t) \end{bmatrix}$$

Ex. : Consider an nth order linear differiantial equation system :

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = bu(t).$$

Assume that $a_n = 1$ without loss of generality and let us define

$$x_1 := y, \; x_2 := \frac{dy}{dt}, \; \cdots \; x_n := \frac{d^{n-1}y}{dt^{n-1}}.$$

Then, we have

$$\dot{x}_1 = x_2, \; \dot{x}_2 = x_3, \; \cdots \; \dot{x}_n = \frac{d^n y}{dt^n}.$$

Since

$$\frac{d^n y}{dt^n} = -a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} - \cdots - a_1\frac{dy}{dt} - a_0 y$$

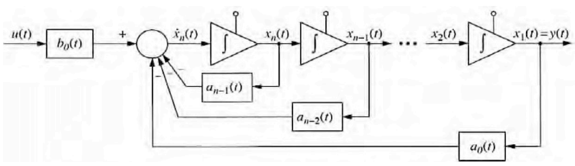
We can write that

$$\frac{d^n y}{dt^n} = -a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} - \cdots - a_1\frac{dy}{dt} - a_0 y,$$

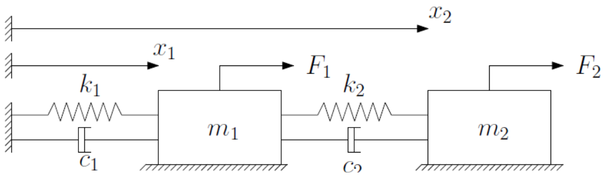
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u,$$

$$x := \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T.$$

$$x(t_o) = \begin{bmatrix} y(t_o) \\ \frac{dy}{dt}(t_o) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(t_o) \end{bmatrix}$$



Ex. : Suppose we have a two degree of spring-mass-damper system with the forces of $f_1(t)$ and $f_2(t)$ acting on the masses shown below



The differential equations of motion can be given as

$$m_1\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 - k_2(x_1 - x_2) - c_2(\dot{x}_1 - \dot{x}_2) + F_1$$
$$m_2\ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) + F_2,$$

where x_1 and x_2 denote the absolute positions of m_1 and m_2. ($x = [x_1\ x_2\ \dot{x}_1\ \dot{x}_2]^T$; $u = [F_1\ F_2]^T$)

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_1 & -(c_1+c_2)/m_1 & c_2/m_1 \\ k_2/m_2 & -k_2/m_2 & c_2/m_1 & -c_2/m_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix} u$$

Assume that the accelerations of the two masses are measured, we get the output as

$$y = \begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_1 & -(c_1+c_2)/m_1 & c_2/m_1 \\ k_2/m_2 & -k_2/m_2 & c_2/m_1 & -c_2/m_2 \end{bmatrix} x + \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix} u$$

State Transformation

We can use different states in order to describe the same dynamical system as the fact that state vectors are not unique to describe a linear system. Assume that one wants to obtain equations for the any system in the examples using a new state vector as $z \triangleq Tx$ for some invertible $T \in \mathbb{R}^{n\times n}$. Then, we immediately have $x = T^{-1}z$, so that $\dot{x} = T^{-1}\dot{z}$. We obtain

$$T^{-1}\dot{z} = AT^{-1}z + Bu$$
$$y = CT^{-1}z + Du.$$

By pre-multiplying the first equation by T, we get the equivalent system description

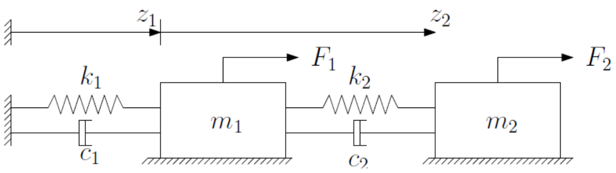
$$\dot{z} = \tilde{A}z + \tilde{B}u$$
$$y = \tilde{C}z + \tilde{D}u,$$

where

$$\tilde{A} := TAT^{-1}, \quad \tilde{B} := TB, \quad \tilde{C} := CT^{-1} \quad \text{and} \quad \tilde{D} := D.$$

Note that, the direct feedforward matrix D is unchanged in the two representations. By the state transformation (or commonly used as coordinate transformation) the only internal description of the system has been changed, but the essential input/output description has not been changed.

Ex. : State transformation can be given physical interpretations. To Show this let us to consider again two degree of spring-mass-damper system with the forces of $f_1(t)$ and $f_2(t)$ acting on the masses shown below. Let us to use the relative positions z_1 and z_2 in deriving the equations of motion, that is $z_1 \triangleq x_1$ and $z_2 \triangleq x_2 - x_1$.(Note that: $z_1 + z_2 = x_2 \Longrightarrow \ddot{x_2} = m_2(\ddot{z_1} + \ddot{z_2})$)



We obtain

$$m_1\ddot{z}_1 = -k_1 z_1 + k_2 z_2 - c_1\dot{z}_1 + c_2\dot{z}_2 + F_1$$

$$m_2(\ddot{z}_1 + \ddot{z}_2) = -k_2 z_2 - c_2\dot{z}_2 + F_2$$

And let us define z $\triangleq [z_1\ z_2\ \dot{z}_1\ \dot{z}_2]^T$. Hence, the state-space representation of the system:

$$\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1/m_1 & k_2/m_1 & -c_1/m_1 & c_2/m_1 \\ k_1/m_1 & -k_2(1/m_1 + 1/m_2) & c_1/m_1 & -c_2(1/m_1 + 1/m_2) \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ -1/m_1 & 1/m_2 \end{bmatrix} F$$

Note that, same result has also been obtained by state transformation as follows:

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} x$$

Linearization

A linear state equation is useful as an approximation to a nonlinear state equation in the following sense

$$\dot{x}(t) = f(x(t), u(t), t); x(t_0) = x_0$$
$$\dot{x}_i = f_i(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t); t\ ), x_i(t_0) = x_{io}$$

for $i = 1, \dots, n$. Suppose $(\dot{x}(t) = f(x(t), u(t), t))$ has been solved for a particular input signal which is called as nominal input $\tilde{u}(t)$, and a particular initial state called as the nominal initial state, $\tilde{x}(t)$. And, let us to interest of the behaviour of the nonlinear state equation for an input and initial state that are close to the nominal values. Namely, u(t) $= \tilde{u}(t) + u_\delta$ and $x_0 = \tilde{x}_0 + x_{0\delta}$ where $\|x_{0\delta}\|$ and $[\![u_\delta]\!]$ are appropriately small for $t \geq t_0$. So that we assume the corresponding solution remains close to $\tilde{x}(t)$, at each t and it can be written $x(t) = \tilde{x}(t) + x_\delta(t)$. Hence, the notation can be given as

$$\frac{d}{dt}\tilde{x}(t) + \frac{d}{dt}x_\delta(t) = \text{f}(\tilde{x}(t) + x_\delta(t), \tilde{u}(t) + u_\delta(\text{t}), \text{t})$$

Assuming the derivatives exists, we can expand the right side using Taylor series about $\tilde{x}(t)$ and $\tilde{u}(t)$ and then retain only the terms through first order. This provides a reasonable approximation since u_δ(t) are assumed $x_\delta(t)$ are assumed to be small for all t :

$$f_i(\tilde{x} + x_\delta, \tilde{u} + u_\delta, t) \approx f_i(\tilde{x}, \tilde{u}, t) + \frac{\partial f_i}{\partial x_1}(\tilde{x}, \tilde{u}, t)x_{\delta 1} + \cdots + \frac{\partial f_i}{\partial x_n}(\tilde{x}, \tilde{u}, t)x_{\delta n}$$
$$+ \frac{\partial f_i}{\partial u_1}(\tilde{x}, \tilde{u}, t)u_{\delta 1} + \cdots + \frac{\partial f_i}{\partial u_m}(\tilde{x}, \tilde{u}, t)u_{\delta m}$$

As performing this expansion for $i = 1, .., n$ and rearranging into vector matrix form gives

$$\frac{d}{dt}\tilde{x}(t) + \frac{d}{dt}x_\delta(t) \approx f(\tilde{x}(t), \tilde{u}(t), t) + \frac{\partial f}{\partial x}(\tilde{x}(t), \tilde{u}(t), t)\ x_\delta(t)$$
$$+ \frac{\partial f}{\partial u}(\tilde{x}(t), \tilde{u}(t), t)\ u_\delta(t)$$

where $\partial f/\partial x$ denotes the Jacobian($\partial fi/\partial xj$). Since

$$\frac{d}{dt}\tilde{x}(t) = f(\tilde{x}(t), \tilde{u}(t), t), \quad \tilde{x}(t_o) = \tilde{x}_o$$

The relation between $x_\delta(t)$ and $u_\delta(t)$ is approximately described by a time varying linear state equation of the form

$$\dot{x}_\delta(t) = A(t)x_\delta(t) + B(t)u_\delta(t), \quad x_\delta(t_o) = x_o - \tilde{x}_o$$

where A(t) and B(t) are the matrices of partial derivatives evaluated using the nominal trajectory data:

$$A(t) = \frac{\partial f}{\partial x}(\tilde{x}(t), \tilde{u}(t), t), \quad B(t) = \frac{\partial f}{\partial u}(\tilde{x}(t), \tilde{u}(t), t)$$

If there is a nonlinear equation

$$y(t) = h(x(t), u(t), t)$$

the function $h(x, u, t)$ can be expanded about $x = \tilde{x}(t)$ and u= $\tilde{u}(t)$ in a similar fashion, after dropping higher-order terms, the approximate description

$$y_\delta(t) = C(t)x_\delta(t) + D(t)u_\delta(t)$$

where $y_\delta(t) = y(t) - \tilde{y}(t)$ and $\tilde{y}(t) = h(\tilde{x}(t), \tilde{u}(t), t)$.

$$C(t) = \frac{\partial h}{\partial x}(\tilde{x}(t), \tilde{u}(t), t), \quad D(t) = \frac{\partial h}{\partial u}(\tilde{x}(t), \tilde{u}(t), t)$$

Ex.(Lin.)

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