MAT2172 Differential Equations II 4.3.2024

Series solutions near to Regular Singular points

$$\left.\begin{aligned} y'' - xy &= 0 \\ xy'' + y &= 0 \end{aligned}\right\}$$ Both are linear, second order variable coefficient dif. eqns.

But, $x = x_0 = 0$ is the ordinary point for $y'' - xy = 0$

$x_0 = 0$ is the singular point for $xy'' + y = 0$

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

x_0 → centered point

Centered points

- Ordinary
- Singular
  - Regular Singular
  - irregular singular points

Definition: $P(x) y'' + Q(x) y' + R(x) y = 0$ --- $(*)$

If $P(x_0) = 0$ for some points x_0,

(or in general $\lim\limits_{x \to x_0} P(x) = 0$)

then, x_0 is a singular point of Eqn (*).

Let divide both sides of the DE (*) by $P(x)$

then, $$1 \cdot y'' + \underbrace{\frac{Q(x)}{P(x)}}_{:=p(x)} y' + \underbrace{\frac{R(x)}{P(x)}}_{:=q(x)} y = 0$$

If,

$$\Rightarrow \lim_{x \to x_0} (x-x_0) \cdot \frac{Q(x)}{P(x)} = \lim_{x \to x_0} (x-x_0) \cdot p(x) = a$$

and

$$\Rightarrow \lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to x_0} (x-x_0)^2 \cdot q(x) = b$$

are both finite, then the point x_0 is regular singular point,

otherwise it is irregular singular point.

Ex: Find all singular points of

$$\underbrace{x^2(1-x)}_{P(x)} y'' + \underbrace{(x-2)}_{Q(x)} y' \underbrace{- 3x}_{R(x)} y = 0$$ and determine whether each one is regular singular point or irr. sing. point.

Let us divide both sides by $P(x)$,

$$\Rightarrow y'' + \boxed{\frac{(x-2)}{x^2.(1-x)}} y' \boxed{- \frac{3x}{x^2(1-x)}} y = 0$$

$p(x)$ $q(x)$

$\Rightarrow P(x_0) = 0 \Rightarrow x_0$ is singular point.

$P(x) = x^2.(1-x) \rightarrow P(x_0) = 0 \Rightarrow x_0^2.(1-x_0) = 0$

$\Rightarrow x_0 = 0, \quad x_0 = 1$

0, 1 are singular points.

At $x_0 = 0$,

$$\lim_{x\to 0} (x-0).p(x) = \lim_{x\to 0} \cancel{x}. \frac{(x-2)}{x^{\cancel{2}}.(1-x)} = \infty \rightarrow$$

not analytic at $x_0 = 0$

$$\lim_{x\to 0} (x-0)^2.q(x) = \lim_{x\to 0} \cancel{x^2}. \frac{-3x}{\cancel{x^2}.(1-x)} = 0$$

Hence, $x_0 = 0$ is the irregular sing. pt.

At $x_0 = 1$,

$$\lim_{x\to 1} \cancel{(x-1)}^{-1}. \frac{x-2}{x^2.\cancel{(1-x)}} = -1$$

$$\lim_{x\to 1} (x-1)^2. \frac{-3x}{x^2.(1-x)} = 0$$

Both are finite. So they are analytic at point $x_0 = 1$

Hence, $x_0 = 1$ is regular sing. pt.

Ex: Find all singular points of

$$\underbrace{x(x-1)^2(x+2)}_{P(x)}y'' + \underbrace{x^2}_{Q(x)}y' \underbrace{-(x^3+2x-1)}_{R(x)}y = 0$$

and classify them.

$$P(x) = x(x-1)^2(x+2)$$

$P(x_0) = 0 \Rightarrow x_0 = 0,\ x_0 = 1,\ x_0 = -2$ are singular points.

At $x_0 = 0$,

$$\lim_{x\to 0} \cancel{(x-0)} \cdot \frac{x^2}{\cancel{x}(x-1)^2(x+2)} = 0$$

$$\lim_{x\to 0} (x-0)^2 \cdot \frac{-(x^3+2x-1)}{x\cdot(x-1)^2\cdot(x+2)} = 0$$

Both are finite, so analytic at $x_0 = 0$, Hence $x_0 = 0$ is a regular sing. pt.

At $x_0 = 1$

$$\lim_{x\to 1} (x-1)\cdot p(x) = \lim_{x\to 1} \cancel{(x-1)} \cdot \frac{x^2}{x\cdot(x-1)^{\cancel{2}}\cdot(x+2)} = \infty$$

$$p(x) = \frac{Q(x)}{P(x)}, \qquad q(x) = \frac{R(x)}{P(x)}$$

$$\lim_{x\to 1} (x-1)^2\cdot q(x) = \lim_{x\to 1} \cancel{(x-1)^2} \cdot \frac{-(x^3+2x-1)}{x\cdot\cancel{(x-1)^2}\cdot(x+2)} = -\frac{2}{3}$$

$x_0 = 1$ is an irregular singular point.

At $x_0 = -2$

$$\lim_{x\to -2} (x+2)\cdot p(x) = \lim_{x\to -2} \cancel{(x+2)}\cdot \frac{x^2}{x\cdot(x-1)^2\cdot\cancel{(x+2)}} = -\frac{4}{18}$$

$$\lim_{x\to -2} (x+2)^2\cdot q(x) = \lim_{x\to -2} (x+2)^{\cancel{2}}\cdot \frac{-(x^3+2x-1)}{x(x-1)^2\cdot\cancel{(x+2)}} = 0$$

Hence $x_0 = -2$ is a regular sing. pt.

Remark:
$x^2 y'' - 2y = 0$ near to the $x_0 = 0$

$x_0 = 0$ singular pt.

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n$$

BUT! $\rightarrow$ at the end of the solution produre we can observe that it can not be found two linerly independent infinite power series solutions.

Theorem: If $x = x_0$ is a regular singular pt of the diff. eqn $P(x)y'' + Q(x)y' + R(x)y = 0$, then there exists at least one solution of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$$

where r is a constant to be determined.

The series will converge at least on some interval

$0 < x - x_0 < R$.

Ex: Find the general solution of the equation

$$4xy'' + 3y' + 3y = 0 \quad (\text{near to the point } x = 0.)$$

$P(x) = 4x = 0 \Rightarrow x = 0$ is the singular point

$$\Rightarrow y'' + \underbrace{\frac{3}{4x}}_{p(x)} y' + \underbrace{\frac{3}{4x}}_{q(x)} y = 0$$

$$\lim_{x \to 0} (x-0) \cdot p(x) = \frac{3}{4}$$

$$\lim_{x \to 0} (x-0)^2 \cdot q(x) = 0$$

Both are finite, analytic. Hence, $X = 0$, is a regular sing. pt.

Let assume that $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - \overset{0}{\overset{\|}{x_0}})^{n+r}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) \cdot a_n \cdot x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r-1) \cdot (n+r) \cdot a_n \cdot x^{n+r-2}$$

$$4xy'' + 3y' + 3y =$$

$$4x\left(\sum_{n=0}^{\infty}(n+r-1).(n+r).a_n.x^{n+r-2}\right) \rightarrow n+r-1$$

$$+3\left(\sum_{n=0}^{\infty}(n+r).a_n.x^{n+r-1}\right)$$

$$+3\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) = 0.$$

$n+r \Leftrightarrow n+r-1$

$n \Leftrightarrow n-1$

$n=0 \Leftrightarrow n=1$

$$= \sum_{n=0}^{\infty}\Big(4(n+r)(n+r-1)+3(n+r)\Big)a_n x^{n+r-1} + \sum_{n=1}^{\infty}3a_{n-1}x^{n+r-1} = 0$$

For $n=0$

$$\underbrace{\Big(4.r.(r-1)+3r\Big).\overset{\neq 0}{a_0}.\overset{\neq 0}{x^{r-1}}}_{=0} + \underbrace{\sum_{n=1}^{\infty}\Big[\Big(4(n+r)(n+r-1)+3(n+r)\Big)a_n + 3a_{n-1}\Big]x^{n+r-1}}_{=0} = 0$$

** $4r(r-1)+3r=0 \longrightarrow$ indicial equation

** $a_n = \dfrac{-3a_{n-1}}{4(n+r)(n+r-1)+3(n+r)} = -\dfrac{3a_{n-1}}{(n+r)\left[4(n+r)-1\right]} \quad \forall n \geq 1$

Recurrence relation

$$4r(r-1)+3r = 4r^2-r=0 \rightarrow r_1=0,\ r_2=1/4$$

For $r_1 = 0$

Then, the recurrence relation becomes $a_n = -\dfrac{3a_{n-1}}{n(4n-1)}$

$\forall n \geq 1$

Let $n=1 \Rightarrow a_1 = -\dfrac{3a_0}{1.3}$

$n=2 \Rightarrow a_2 = -\dfrac{3a_1}{2.7}$

$n=3 \Rightarrow a_3 = -\dfrac{3a_2}{3.11}$

$n=4 \Rightarrow a_4 = -\dfrac{3a_3}{4.15} = +\dfrac{3.3a_2}{4.15.3.11}$

$= -\dfrac{3.3.3a_1}{4.15.3.11.2.7}$

$= +\dfrac{3.3.3.3a_0}{4.15.3.11.2.7.1.3}$

Let us take $a_0 = 1$,

$$a_n = \frac{(-1)^n . 3^{n-1}}{n!. 7.11.15. \dots (4n-1)}$$

Hence, $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ $=0$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n . 3^{n-1}}{n!. 7.11 \dots (4n-1)} . x^n$$

For $r = 1/4$

Recurrence relation becomes $a_n = -\dfrac{3a_{n-1}}{n(4n+1)} \quad \forall n \geq 1$

Setting $a_0 = 1$, we get

$$a_1 = -\frac{3a_0}{5} = -\frac{3}{5}$$

$$a_2 = -\frac{3a_1}{2.9} = \frac{3^2}{2.5.9}$$

$$a_3 = \frac{-3^3}{2.3.5.9.13}$$

$$a_4 = \frac{3^4}{2.3.4.5.9.13.17}$$

So, $$a_n = \frac{(-1)^n . 3^n}{n! . 5.9.13 \ldots (4n+1)}$$

Hence, $$y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad r = 1/4$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n . 3^n}{n! . 5.9.13. \ldots (4n+1)} x^{n+1/4}$$

$$y(x) = C_1 \underline{\underline{y_1}} + C_2 \underline{\underline{y_2}}$$ inf. power series.

NOTE: $4xy'' + 3y' + 3y = 0$

$P(x) = 4x = 0 \Rightarrow x = 0$ is the singular point

$$\Rightarrow 1y'' + \underbrace{\frac{3}{4x}}_{p(x)}y' + \underbrace{\frac{3}{4x}}_{q(x)}y = 0$$

$$\lim_{x\to 0} (x-0)\cdot p(x) = \frac{3}{4} =: p_0$$

$$\lim_{x\to 0} (x-0)^2\cdot q(x) = 0 =: q_0$$

$$\Rightarrow \boxed{r\cdot(r-1) + p_0 r + q_0 = 0}$$ indicial equation formula.

$$r\cdot(r-1) + \frac{3}{4}r + 0 = 0$$

$$\Rightarrow 4r\cdot(r-1) + 3r = 0 \Rightarrow 4r^2 - r = 0$$

$$\Rightarrow r(4r-1) = 0$$

$$\Rightarrow r_1 = 0,\ r_2 = 1/4$$

Theorem: Consider the dif. eqn

$$x^2y'' + x(xp(x))y' + (x^2q(x))y = 0$$

where $x=0$ is a regular singular pt.

Then $xp(x)$ and $x^2q(x)$ are analytic at $x=0$

with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

for $|x| < \rho$, where $\rho > 0$ is the minimum of

the radii of convergence of the power series

for $xp(x)$ and $x^2q(x)$.

Let r_1 and r_2 are the roots of the

indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

with $r_1 \geq r_2$ if r_1 and r_2 are real.

Then in either the interval $-\rho < x < 0$ or

the interval $0 < x < \rho$,

there exist a solution of the form

$$y_1(x) = |x|^{r_1}\left(1+\sum_{n=1}^{\infty} a_n(r_1)\, x^n\right)$$

where the $a_n(r_1)$ are given by the recurrence relation with $a_0=1$ and $r=r_1$.

Case 1: If r_1-r_2 is not zero or a positive integer, then in either the interval $-\rho<x<0$, or the interval $0<x<\rho$, there exist a solution of the form

$$y_2(x) = |x|^{r_2}\left(1+\sum_{n=1}^{\infty} a_n(r_2)\, x^n\right)$$

The $a_n(r_2)$ are also determined by the recurrence relation with $a_0=1$ are $r=r_2$.

The power series y_1 & y_2 are convergent at least $|x|<\rho$.

Case 2: If $r_1 = r_2$, then the second solution is

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n$$

Case 3: If $r_1 - r_2 = N$, a positive integer, then

$$y_2(x) = a y_1(x)\ln|x| + |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right)$$

The coefficients $a_n(r_1)$, $b_n(r_1)$ and $c_n(r_1)$ and the constant a can be determined by substituting the form of the series solutions for y in the given equation.

If $a = 0$, then there is no logaritmic term in the solution.

For each y_2 in the Case 2 and Case 3 can be said they are convergent at least for $|x| < \rho$ and they defines a function that is analytic in some neighborhood of $x=0$.