On ϕ -1-absorbing prime ideals

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ABSTRACT. In this paper, we introduce ϕ -1-absorbing prime ideals in commutative rings. Let R be a commutative ring with a nonzero identity $1 \neq 0$ and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. A proper ideal I of R is called a ϕ -1-absorbing prime ideal if for each nonunits $x, y, z \in R$ with $xyz \in I - \phi(I)$, then either $xy \in I$ or $z \in I$. In addition to give many properties and characterizations of ϕ -1-absorbing prime ideals, we also determine rings in which every proper ideal is ϕ -1-absorbing prime.

1. Introduction

Throughout the paper, we focus only on commutative rings with a nonzero identity. Let R will always denote such a ring. We will denote the set of all ideals of R by $\mathcal{I}(R)$. A proper ideal I of R is an element $I \in \mathcal{I}(R)$ with $I \neq R$. For many years, numerous types of ideals have been developed such as prime, primary, maximal, etc. All of them play significant role when characterizing a ring. The concept of prime ideals and its generalizations have a significant place in commutative algebra since they are used in understanding the structure of rings. Recall that a proper ideal I of R is said to be a *prime ideal* if whenever $xy \in I$ for some $x, y \in R$, then either $x \in P$ or $y \in P$ [4]. The importance of prime ideals led many researchers to work prime ideals and its generalizations. See, for example, [6], [9] and [13]. In [2], Anderson and Smith introduced a notion of weakly prime ideal which is a generalization of prime ideals. A proper ideal I of Ris called *weakly prime ideal* if $0 \neq xy \in I$ for some elements $x, y \in I$ implies that $x \in I$ or $y \in I$. They gave many results concerning weakly prime ideals and used it to study factorization in commutative rings with zero divisors. Also, they gave necessary and sufficient conditions so that any proper ideal of R can be written as a product of weakly prime ideals. It is clear that every prime ideal is weakly prime but the converse is not true in general. Afterwards, Badawi, in his celebrated paper [5], introduced the notion of 2-absorbing ideals and used them to characterize Dedekind domains. Recall from [5], that a nonzero proper ideal I of R is called 2-absorbing ideal if $xyz \in I$ for some $x, y, z \in R$ implies either $xy \in I$ or $xz \in I$ or $yz \in I$. Note that every prime ideal is also a 2-absorbing ideal. After this, over the past decades, 2-absorbing version of ideals and many generalizations of 2-absorbing ideals attracted considerable attention by many researchers in [8], [15] and [14].

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In [10], in order to study unique factorization domain, Bhatwadekar and Sharma defined almost prime ideals which is a generalization of prime ideals. A proper ideal I is called almost prime ideal if $xy \in I - I^2$ for some $x, y \in R$ implies that $x \in I$ or $y \in I$. They investigated the relations among the prime ideals, pseudo prime ideals and almost prime ideals of R. Badawi and Darani in [7] defined and studied weakly 2-absorbing ideals which is a generalization of weakly prime ideals. A proper ideal I of R is called a *weakly 2-absorbing ideal* if for each $x, y, z \in R$ with $0 \neq xyz \in I$, then either we have $xy \in I$ or $xz \in I$ or $yz \in I$. In [1], Anderson and Bataineh defined a new class of prime ideals. A proper ideal I of R is called ϕ -prime ideal if whenever $xy \in I - \phi(I)$ for some $x, y \in R$ then either $x \in I$ or $y \in I$ where $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ is a function. They showed that a prime ideal and a ϕ -prime ideal have some similar properties. Recently, in [19], Yassine et al. introduced 1-absorbing prime ideal. This type of ideals is a generalization of prime ideals. A proper ideal I of R is called 1-absorbing prime ideal if whenever $xyz \in I$ for some nonunits $x, y, z \in R$ then either $xy \in I$ or $z \in I$. Note that every prime ideal is 1-absorbing prime and every 1-absorbing prime ideal is 2-absorbing ideal. The converses are not true. For instance, $P = 6\mathbb{Z}$ is a 2-absorbing ideal of \mathbb{Z} but not a 1-absorbing prime ideal and also $P = (\overline{0})$ is a 1-absorbing prime ideal of \mathbb{Z}_4 which is not prime. They characterized 1-absorbing prime ideals of some special rings such as valuation domain and principal ideal domain. They also gave Prime Avoidance Theorem for 1-absorbing prime ideals. More currently, Koc et al. defined weakly-1-absorbing prime ideals which is a generalization of 1-absorbing prime ideal [17]. A proper ideal I of R is called weakly-1-absorbing prime ideal if $0 \neq xyz \in I$ for some nonunits $x, y, z \in R$ implies that $xy \in I$ or $z \in I$. They gave many properties of this class of ideals and characterized rings that every proper ideal is weakly-1-absorbing ideal. Moreover, they investigated weakly-1-absorbing ideal in C(X), which is the set of all real-valued continuous functions of topological space X.

In this paper, we define ϕ -1-absorbing prime ideals as a new class of ideals which is generalization of 1-absorbing prime ideals. A proper ideal I of R is called ϕ -1-absorbing prime ideal if whenever $xyz \in I - \phi(I)$ for some nonunits $x, y, z \in R$ then $xy \in I$ or $z \in I$. Among other results in this paper, we give some relations between ϕ -1-absorbing prime ideals and other classical ideals such as weakly prime ideals, ϕ -prime ideals, 1-absorbing prime ideals and weakly 1-absorbing prime ideals (See, Proposition 1). In particular, we show that every ϕ -prime ideal is also a ϕ -1-absorbing prime ideal. But the converse is not true in general (See, Example 4). Hovewer, we give a condition under which any ϕ -1-absorbing prime ideal is ϕ -prime (See, Theorem 4). Also, we give some characterizations of ϕ -1-absorbing prime ideals in general rings, in factor ring, in localization of rings, in cartesian product of rings (See, Theorem 1, Theorem 5, Theorem 6, Theorem 9). Finally, we determine rings over which every ideal is almost 1-absorbing prime ideal (See, Theorem 10).

2. Characterization of ϕ -1-absorbing prime ideals

Let R be a commutative ring. Define a function $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$. This function maps an ideal of R to an ideal of R or \emptyset .

DEFINITION 1. Let R be a ring and I be a proper ideal of R. I is called ϕ -1absorbing prime ideal of R if whenever $xyz \in I - \phi(I)$ for some nonunits $x, y, z \in R$ then $xy \in I$ or $z \in I$.

The following notations will be used for the rest of the paper.

EXAMPLE 1. Let R be a commutative ring and $\phi_{\alpha} : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function. The following gives types of 1-absorbing prime ideals corresponding to ϕ_{α} .

ϕ_{\emptyset}	$\phi(I) = \emptyset$	1-absorbing prime ideal
ϕ_0	$\phi(I) = 0$	weakly-1-absorbing prime ideal
ϕ_2	$\phi(I) = I^2$	almost-1-absorbing prime ideal
ϕ_n	$\phi(I) = I^n$	n-almost-1-absorbing prime ideal
ϕ_w	$\phi(I) = \bigcap_{n=1}^{\infty} I^n$	w-1-absorbing prime ideal
ϕ_1	$\phi(I) = I$	any ideal

Consider two functions $\phi, \psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$. Then $\phi \leq \psi$ if $\phi(I) \subseteq \psi(I)$ for all ideals of R. Moreover, note that $\phi_{\emptyset} \leq \phi_0 \leq \phi_w \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

We will assume that $\phi(I) \subseteq I$ throughout the paper.

PROPOSITION 1. (i) Let I be a proper ideal of R and $\phi, \psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions with $\phi \leq \psi$. If I is a ϕ -1-absorbing prime ideal, then I is a ψ -1-absorbing ideal.

(ii) I is a 1-absorbing prime ideal \Rightarrow I is a weakly 1-absorbing prime ideal \Rightarrow I is a w-1-absorbing prime ideal \Rightarrow I is an n-almost 1-absorbing prime ideal for each $n \ge 2 \Rightarrow$ I is an almost 1-absorbing prime ideal.

(iii) I is an n-almost 1-absorbing prime ideal if and only for each $n \ge 2$ if and only if I is a w-1-absorbing prime ideal.

(iv) Every ϕ -prime ideal is a ϕ -1-absorbing prime ideal.

PROOF. (i): Assume that I is a ϕ -1-absorbing prime ideal. Let $xyz \in I - \psi(I)$ for some nonunits $x, y, z \in R$. Then, $xyz \in I - \phi(I)$ and since I is ϕ -1-absorbing ideal, $xy \in I$ or $z \in I$ which completes the proof.

(ii): Follows from the fact that $\phi_{\emptyset} \leq \phi_0 \leq \phi_w \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \phi_2$ and (i).

(iii): By (ii), we know that if I is a w-1-absorbing prime ideal, then I is an n-almost 1-absorbing prime ideal for each $n \ge 2$. Now, assume that I is an n-almost 1-absorbing prime ideal if and only for each $n \ge 2$. Let $xyz \in I - \bigcap_{n=1}^{\infty} I^n$ for some nonunits $x, y, z \in R$. Then there exists $m \ge 2$ such that $xyz \notin I^m$. Since I is an m-almost 1-absorbing prime ideal of R and $xyz \in I - I^m$, then either we have $xy \in I$ or $z \in I$.

(iv): It is clear.

EXAMPLE 2. (weakly 1-absorbing prime ideal that is not 1-absorbing prime ideal) Let p, q be distinct prime numbers and consider the ring $R = \mathbb{Z}_{pq^2}$. Then $I = (\overline{0})$ is a weakly 1-absorbing prime ideal of R. Since $\overline{pqq} \in I$ and $\overline{pq}, \overline{q} \notin I$, I is not a 1-absorbing prime ideal of R.

EXAMPLE 3. (w-1-absorbing prime ideal that is not weakly 1-absorbing prime ideal) Let I be an idempotent ideal of R, that is, $I = I^2$. Then I is a w-1-absorbing prime ideal since $I^n = I$ for each $n \ge 2$. But I may not be a weakly 1-absorbing prime ideal of R. For instance, take $R = \mathbb{Z}_2^4$ and $I = \mathbb{Z}_2 \times (0) \times$ (0) × (0). Then I is a w-1-absorbing prime ideal since it is idempotent. Now, take the nonunits x = (1, 1, 1, 0), y = (1, 1, 0, 1) and y = (1, 0, 1, 1) in R. Then $0 \neq xyz \in I$ but $xy, z \notin I$. So it follows that I is not a weakly 1-absorbing prime ideal of R.

EXAMPLE 4. $(\phi$ -1-absorbing prime ideal that is not ϕ -prime) Take R as in Example 2 and consider the ideal $I = (\overline{q^2})$ of R. Suppose that $\phi(I) = (\overline{0})$. Then I is not ϕ -prime since $\overline{qq} \in I - \phi(I)$ and $\overline{q} \notin I$. Now, take nonunits $\overline{x}, \overline{y}, \overline{z} \in R$ such that $\overline{0} \neq \overline{xyz} \in I$. Then it is clear that $q^2|xyz$ and $pq^2 \nmid xyz$. If $q^2|xy$, then we are done. So assume that $q^2 \nmid xy$. On the other hand, since $q^2|xyz$, we have q|z. If $q^2|z$, again we are done. So we may assume that q|z but $q^2 \nmid z$. Since $q^2|xyz$, q|z and $q^2 \nmid z$, we have either q|x or q|y. Without loss of generality, suppose that q|x but $q \nmid$ y. Since \overline{y} is not unit, we have p|y and in this case $\overline{xyz} = \overline{0}$ which is a contradiction. Therefore, we have either $q^2|xy$ or $q^2|z$, namely, $xy \in I$ or $z \in I$.

THEOREM 1. Let R be a commutative ring and I a proper ideal of R. The following statements are equivalent.

(i) I is a ϕ -1-absorbing prime ideal of R.

(ii) For each nonunits $x, y \in R$ with $xy \notin I$ implies $(I : xy) = I \cup (\phi(I) : xy)$.

(iii) For each nonunits $x, y \in R$ with $xy \notin I$ gives either (I : xy) = I or $(I : xy) = (\phi(I) : xy)$.

(iv) For each nonunits $x, y \in R$ and proper ideal J of R such that $xyJ \subseteq I$ but $xyJ \not\subseteq \phi(I)$ implies either $xy \in I$ or $J \subseteq I$.

(v) For each nonunit $x \in R$ and proper ideals J, K of R such that $xJK \subseteq I$ but $xJK \not\subseteq \phi(I)$, either $xJ \subseteq I$ or $K \subseteq I$.

(vi) For each proper ideals J, K, L of R such that $JKL \subseteq I$ but $JKL \not\subseteq \phi(I)$, either $JK \subseteq I$ or $L \subseteq I$.

PROOF. $(i) \Rightarrow (ii)$: Assume that I is a ϕ -1-absorbing ideal of R and $xy \notin I$ for some nonunit elements $x, y \in R$. It is clear that $I \cup (\phi(I) : xy) \subseteq (I : xy)$. On the other hand, choose $z \in (I : xy)$ and so $xyz \in I$. If $xyz \notin \phi(I)$, then $z \in I$. Now suppose $xyz \in \phi(I)$. Then, $z \in (\phi(I) : xy)$. Therefore, it gives $(I : xy) \subseteq I \cup (\phi(I) : xy)$.

 $(ii) \Rightarrow (iii)$: Since $(I : xy) = I \cup (\phi(I) : xy)$, (I : xy) must be one of the component in the union.

 $(iii) \Rightarrow (iv)$: Assume that $xyJ \subseteq I$ but $xyJ \not\subseteq \phi(I)$. Let $xy \notin I$. Then, either $(I : xy) = (\phi(I) : xy)$ or (I : xy) = I by (iii). Suppose the former case holds. Since $xyJ \subseteq I$, we have $J \subseteq (I : xy) = (\phi(I) : xy)$. It gives $xyJ \subseteq \phi(I)$ which is a contradiction. Now, suppose the latter case holds. Then, $J \subseteq (I : xy) = I$ showing $J \subseteq I$, as needed.

 $(iv) \Rightarrow (v)$: Let $xJK \subseteq I$ and $xJK \not\subseteq \phi(I)$. Suppose $xJ \not\subseteq I$ and $K \not\subseteq I$. Then there exists $a \in J$ such that $xa \notin I$. Also, since $xJK \not\subseteq \phi(I)$ there exists $b \in J$ such that $xbK \not\subseteq \phi(I)$. Now assume that $xaK \notin \phi(I)$. Since x, a are nonunits and $xaK \subseteq I$, we have either $xa \in I$ or $K \subseteq I$, a contradiction. So, we get $xaK \in \phi(I)$. Also, we have $x(a+b)K \subseteq I - \phi(I)$ and it implies $x(a+b) \in I$. Since $xbK \subseteq I - \phi(I)$ and $K \not\subseteq I$, we get $xb \in I$. Thus, we obtain $xa \in I$ giving a contradiction. This proves $xJ \subseteq I$ or $K \subseteq I$.

 $(v) \Rightarrow (vi)$: Let $JKL \subseteq I$ but $JKL \not\subseteq \phi(I)$ for some proper ideals J, K and L of R. Assume that $JK \not\subseteq I$ and $L \not\subseteq I$. Then, there exists $y \in J$ such that $yK \not\subseteq I$. Also since $JKL \not\subseteq \phi(I), xKL \notin \phi(I)$ for some $x \in J$. Then, we get $xK \subseteq I$ since $xKL \subseteq I - \phi(I)$. Suppose $yKL \not\subseteq \phi(I)$. By (v), this gives $yK \subseteq I$ or $L \subseteq I$, which is contradiction. So, $yKL \subseteq \phi(I)$. As $(x+y)KL \subseteq I - \phi(I)$, we have $(x+y)K \subseteq I$. This implies $yK \subseteq I$, a contradiction.

 $(vi) \Rightarrow (i)$: Let $xyz \in I - \phi(I)$. Then, $(x)(y)(z) \subseteq I$ and $(x)(y)(z) \not\subseteq \phi(I)$. Hence, $(x)(y) \subseteq I$ or $(z) \subseteq I$ showing that $xy \in I$ or $z \in I$, as desired. \Box

DEFINITION 2. Let I be a ϕ -1-absorbing prime ideal and x, y, z be nonunit elements of R. If $xyz \in \phi(I)$, $xy \notin I$ and $z \notin I$, then we say that (x, y, z) is a ϕ -1-triple zero of I.

REMARK 1. (i) Let I be a ϕ -1-absorbing prime ideal of R. Then I has a ϕ -1triple zero if and only if there exists $z \notin I$ and a nonunit element $y \in R$ such that $(\phi(I) : yz) \nsubseteq (I : y)$.

(ii) Let I be a proper ideal of R. Then I is a 1-absorbing prime ideal if and only if the following two conditions must be hold:

(a) I is a ϕ -1-absorbing prime ideal of R.

(b) For each $z \notin I$ and nonunit element $y \in R$, we have $(\phi(I) : yz) \subseteq (I : y)$.

THEOREM 2. Suppose that I is a ϕ -1-absorbing prime ideal of R that is not 1-absorbing prime and (x, y, z) is a ϕ -1-triple zero of I. Then,

(i) $xyI \subseteq \phi(I)$.

(ii) If $xz, yz \notin I$, then xzI, yzI, xI^2 , yI^2 , $zI^2 \subseteq \phi(I)$. In particular, $I^3 \subseteq \phi(I)$.

PROOF. (i): Let I be a ϕ -1-absorbing prime ideal of R that is not 1-absorbing prime and (x, y, z) be a ϕ -1-triple zero of I. Then we have $xyz \in \phi(I)$, $xy \notin I$ and $z \notin I$. Suppose $xyI \not\subseteq \phi(I)$. Then, there exists $a \in I$ such that $xya \notin \phi(I)$. So, $xy(z+a) \notin I - \phi(I)$. If z + a is unit, then $xy \in I$, a contradiction. Now assume that z + a is nonunit and so we get $xy \in I$ or $z \in I$, again a contradiction. Thus, we have $xyI \subseteq \phi(I)$.

(ii): Now, assume that $xz, yz \notin I$. We will show that $xzI, yzI \subseteq \phi(I)$. Suppose that $xzI \not\subseteq \phi(I)$. Then there exists an element $a \in I$ such that $xza \notin \phi(I)$. This implies that $x(y+a)z \in I - \phi(I)$. If y+a is unit, then $xz \in I$ which is a contradiction. Thus y + a is nonunit. Since I is a ϕ -1-absorbing prime ideal, we conclude either $x(y+a) \in I$ or $z \in I$, which implies $xy \in I$ or $z \in I$, again a contradiction. Thus, $xzI \subseteq \phi(I)$. By using similar argument, we have $yzI \subseteq \phi(I)$. Now, we will show that $xI^2 \subseteq \phi(I)$. Suppose to the contrary. Then, there exists $a, b \in I$ such that $xab \notin \phi(I)$. It implies $x(y+a)(z+b) \in I - \phi(I)$. If (y+a) is unit, $x(z+b) \in I$ which gives $xz \in I$, a contradiction. Similarly, (z+b) is nonunit. Then, either $x(y+a) \in I$ or $z+b \in I$ implying that $xy \in I$ or $z \in I$. Thus, we have $xI^2 \subseteq \phi(I)$. Similarly, we get $yI^2 \subseteq \phi(I)$ and $zI^2 \subseteq \phi(I)$, we are done. For the rest, if $I^3 \not\subseteq \phi(I)$, there exists $a, b, c \in I$ such that $abc \notin \phi(I)$. Then, $(x+a)(y+b)(z+c) \in I - \phi(I)$. If x+a is unit, then we obtain $(y+b)(z+c) = yz + yc + zb + bc \in I$ and so $yz \in I$, which is a contradiction. Similarly, we can show that y + b and z + c are nonunits. Then, we get $(x+a)(y+b) \in I$ or $z+c \in I$. This gives $xy \in I$ or $z \in I$, again a contradiction. Hence, $I^3 \subseteq \phi(I)$.

THEOREM 3. Let R be a ring and a be a nonunit element of R. Suppose that $(0:a) \subseteq (a)$ (e.g., a is regular). Then, (a) is ϕ -1-absorbing prime ideal with $\phi \leq \phi_2$ if and only if (a) is a 1-absorbing prime ideal.

PROOF. If (a) is 1-absorbing prime ideal, then (a) is ϕ -1-absorbing ideal. For the other direction, assume that (a) is ϕ -1-absorbing prime ideal with $\phi \leq \phi_2$. Then, it is also ϕ_2 -1-absorbing prime ideal by Proposition 1. Let $xyz \in (a)$ for some nonunits $x, y, z \in R$. If $xyz \notin (a)^2$, then $xy \in (a)$ or $z \in (a)$. So suppose $xyz \in (a)^2$. We have $xy(z + a) \in (a)$. If z + a is unit, we are done. Hence, we can assume z + a is nonunit. Assume that $xy(z + a) \notin (a)^2$. Then we get either $xy \in (a)$ or $z + a \in (a)$ implying $xy \in (a)$ or $z \in (a)$. Now assume $xy(z + a) \in (a)^2$. This gives $ayz \in (a)^2$ and so there exists $t \in R$ such that $ayz = a^2t$. Thus we have $yz - at \in (0 : a) \subseteq (a)$. Therefore, $yz \in (a) + (0 : a) \subseteq (a)$, as needed. \Box

Now, we give a condition for a ϕ -1-absorbing prime ideal of R to become a ϕ -prime ideal of R.

THEOREM 4. Let I be a proper ideal of a non-quasi local ring R. Suppose that $(\phi(I) : x)$ is not maximal ideal for each $x \in I$. The following statements are equivalent.

(i) I is a ϕ -prime ideal of R.

(ii) I is a ϕ -1-absorbing prime ideal of R.

PROOF. $(i) \Rightarrow (ii)$: Follows from Proposition 1.

 $(ii) \Rightarrow (i)$: Let I be a ϕ -1-absorbing prime ideal of R. Choose $x, y \in R$ such that $xy \in I - \phi(I)$. If x or y is unit, then $x \in I$ or $y \in I$ which is needed. So suppose that x, y are nonunits in R. Since $xy \notin \phi(I)$, $(\phi(I) : xy)$ is proper. Choose a maximal ideal \mathfrak{m}_1 of R containing $(\phi(I): xy) \subseteq \mathfrak{m}_1$. Since R is non-quasi-local, there exists a different maximal ideal \mathfrak{m}_2 of R. Now, take $z \in \mathfrak{m}_2 - \mathfrak{m}_1$. Then $z \notin (\phi(I) : xy)$, and so we have $(zx)y \in I - \phi(I)$. Since I is a ϕ -1-absorbing prime ideal of R, we get either $zx \in I$ or $y \in I$. If $y \in I$, then we are done. So assume that $zx \in I$. As $z \notin \mathfrak{m}_1$, then there exists an $a \in R$ such that $1 + az \in \mathfrak{m}_1$. Note that 1 + az is nonunit. If $1 + az \notin (\phi(I) : xy)$, then we have $(1 + az)xy \in I - \phi(I)$ implying $(1+az)x \in I$ and so $x \in I$ since $zx \in I$. So assume that $1+az \in (\phi(I):xy)$, that is, $xy(1+az) \in \phi(I)$. Now, choose an element $b \in \mathfrak{m}_1 - (\phi(I) : xy)$. Then we have $(1 + az + b)xy \in I - \phi(I)$. On the other hand, since $1 + az + b \in \mathfrak{m}_1$, 1 + az + bis nonunit. This implies that $(1 + az + b)x \in I$. Also, since $bxy \in I - \phi(I)$, we get $bx \in I$. Then we have $x = (1 + az + b)x - a(zx) - bx \in I$. Therefore, I is a ϕ -prime ideal of R. \square

Now, for any ideal J of R define a function $\phi_J : \mathcal{I}(R/J) \to \mathcal{I}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J))/J$ where $J \subseteq I$ and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$. Also, note that $\phi_J(I/J) \subseteq I/J$.

THEOREM 5. (i) Let I be a ϕ -1-absorbing prime ideal of R. Then $I/\phi(I)$ is a weakly 1-absorbing prime ideal of $R/\phi(I)$.

(ii) Let $I/\phi(I)$ be a weakly 1-absorbing prime ideal of $R/\phi(I)$ and $u(R/\phi(I)) = \{x + \phi(I) : x \in u(R)\}$. Then I is a ϕ -1-absorbing prime ideal of R.

(iii) Let I, J be two ideals of R with $J \subseteq I$ and I be a ϕ -1-absorbing prime ideal. Then, I/J is a ϕ_J -1-absorbing prime ideal of R/J.

PROOF. (i): Let $\overline{0} \neq \overline{x}\overline{y}\overline{z} \in I/\phi(I)$ for some nonunits $\overline{x}, \overline{y}, \overline{z} \in R/\phi(I)$, where $\overline{x} = x + \phi(I), \overline{y} = y + \phi(I)$ and $\overline{z} = z + \phi(I)$. Then x, y, z are nonunits in R and $xyz \in I - \phi(I)$. Since I is a ϕ -1-absorbing prime ideal of $R, xy \in I$ or $z \in I$. Then, we get $\overline{xy} \in I/J$ or $\overline{z} \in I/J$ which completes the proof.

(ii): Let $I/\phi(I)$ be a weakly 1-absorbing prime ideal of $R/\phi(I)$ and $u(R/\phi(I)) = \{x + \phi(I) : x \in u(R)\}$. Choose nonunits x, y, z in R such that $xyz \in I - \phi(I)$. Then we have $\overline{0} \neq \overline{x}\overline{y}\overline{z} \in I/\phi(I)$. Since $u(R/\phi(I)) = \{x + \phi(I) : x \in u(R)\}$, $\overline{x}, \overline{y}$ and \overline{z} are nonunits in $R/\phi(I)$. Since $I/\phi(I)$ is a weakly 1-absorbing prime ideal, we have either $\overline{x}\overline{y} \in I/\phi(I)$ or $\overline{z} \in I/\phi(I)$, which implies $xy \in I$ or $z \in I$. Therefore, I is a ϕ -1-absorbing prime ideal of R.

(iii): It is similar to (i)

Let R be a commutative ring and S be a multiplicatively closed subset of R. Consider the function $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$. Define $\phi_S : \mathcal{I}(S^{-1}R) \to \mathcal{I}(S^{-1}R) \cup \{\emptyset\}$ by $\phi_S(I) = S^{-1}\phi(I \cap R)$ and $\phi_S(I) = \emptyset$ if $\phi(I \cap R) = \emptyset$. Here, it is easy to see that $\phi_S(I) \subseteq I$.

THEOREM 6. Let R be a commutative ring, $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function, I be a ϕ -1-absorbing prime ideal of R and S be a multiplicatively closed subset of R with $I \cap S = \emptyset$ and $S^{-1}\phi(I) \subseteq \phi_S(S^{-1}I)$. Then, $S^{-1}I$ is a ϕ_S -1-absorbing prime ideal of $S^{-1}R$. Furthermore, if $S^{-1}I \neq S^{-1}\phi(I)$, then $S^{-1}I \cap R = I$.

PROOF. Let $\frac{x}{s} \frac{y}{t} \frac{z}{u} \in S^{-1}I - \phi_S(S^{-1}I)$ for some nonunits $\frac{x}{s}, \frac{y}{t}, \frac{z}{u} \in S^{-1}R$. Then, there exists $s' \in S$ such that $s'xyz \in I$ but $s^*xyz \notin \phi(S^{-1}I \cap R)$ for all $s^* \in S$. If $s'xyz \in \phi(I)$, then we have $\frac{x}{s} \frac{y}{t} \frac{z}{t} \in \phi(I)_S \subseteq \phi_S(S^{-1}I)$, a contradiction. So we get $s'xyz = (s'x)yz \in I - \phi(I)$. Since s'x, y, z are nonunits in R and I is a ϕ -1-absorbing prime ideal, we get $s'xy \in I$ or $z \in I$. This implies $\frac{x}{s} \frac{y}{t} = \frac{s'xy}{s'st} \in S^{-1}I$ or $\frac{z}{u} \in S^{-1}I$.

Now we will show that $S^{-1}I \cap R = I$. Let $a \in S^{-1}I$. Then, there exists $s \in S$ such that $sa \in I$. If s is unit, we are done. If a is unit, it contrdicts with $I \cap S = \emptyset$. So we can assume s and a are nonunits in R. If $s^2a = ssa \notin \phi(I)$, we get $s^2 \in I$ or $a \in I$. Since former case is not possible, we have $a \in I$. In the case $s^2a \in \phi(I)$, we have $a \in S^{-1}\phi(I) \cap R$. So we obtain $S^{-1}I \cap R \subseteq I \cup (S^{-1}\phi(I) \cap R)$. Thus, we conclude that either $S^{-1}I \cap R = I$ or $S^{-1}I \cap R = S^{-1}\phi(I) \cap R$. Since latter case contradicts with the assumption, we have $S^{-1}I \cap R = I$.

Let R_1, R_2 be commutative rings and $\phi_1 : \mathcal{I}(R_1) \to \mathcal{I}(R_1) \cup \{\emptyset\}, \phi_2 : \mathcal{I}(R_2) \to \mathcal{I}(R_2) \cup \{\emptyset\}$ be two functions. Suppose that $R = R_1 \times R_2$ and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ is a function defined by $\phi(I_1 \times I_2) = \phi_1(I_1) \times \phi_2(I_2)$ for each ideal I_k of R_k . Then ϕ is denoted by $\phi = \phi_1 \times \phi_2$.

THEOREM 7. Let R_1, R_2 be commutative rings and $\phi_1 : \mathcal{I}(R_1) \to \mathcal{I}(R_1) \cup \{\emptyset\}$, $\phi_2 : \mathcal{I}(R_2) \to \mathcal{I}(R_2) \cup \{\emptyset\}$ be two functions. Suppose that $I = I_1 \times I_2$, where I_i is an ideal of R_i for each i = 1, 2, and $\phi = \phi_1 \times \phi_2$. If $I = I_1 \times I_2$ is a ϕ -1-absorbing prime ideal of R, then one of the following three conditions must be hold.

(i) $\phi(I) = I$.

(ii) $I = I_1 \times R_2$ and I_1 is a ϕ_1 -prime ideal of R_1 which must be prime if $\phi_2(R_2)$ is not unique maximal ideal of R_2 (e.g. R_1, R_2 are not quasi-local).

(iii) $I = R_1 \times I_2$ and I_2 is a ϕ_2 -prime ideal of R_2 which must be prime if $\phi_1(R_1)$ is not unique maximal ideal of R_1 (e.g. R_1, R_2 are not quasi-local).

PROOF. Suppose that I is a ϕ -1-absorbing prime ideal of R. First, we will show that I_1 is a ϕ_1 -prime ideal of R_1 . To see this, choose $x, y \in R$ such that $xy \in I_1 - \phi_1(I_1)$. Then we have $(x, 0)(1, 0)(y, 0) = (xy, 0) \in I - \phi(I)$ for some nonunits $(x, 0), (1, 0), (y, 0) \in R$. Since I is a ϕ -1-absorbing prime ideal of R, we get either $(x, 0)(1, 0) = (x, 0) \in I$ or $(y, 0) \in I$ implying that $x \in I_1$ or $y \in I_1$. Therefore, I_1 is a ϕ_1 -prime ideal of R_1 . Similar argument shows that I_1 is a ϕ_2 -prime ideal of R_2 . Now assume that $\phi(I) \neq I$. Then either $\phi_1(I_1) \neq I_1$ or $\phi_2(I_2) \neq I_2$. Suppose that $\phi_1(I_1) \neq I_1$. Then there exists $x \in I_1 - \phi_1(I_1)$. This implies that $(1, 0)(1, 0)(x, 1) = (x, 0) \in I - \phi(I)$. Then we have either $1 \in I_1$ or $1 \in I_2$, that is, $I_1 = R_1$ or $I_2 = R_2$. Without loss of generality, we may assume that $I_1 = R_1$. Now, we will show that $I = R_1 \times I_2$ and I_2 is prime in R_2 if $\phi_1(R_1)$ is not unique maximal ideal of R_1 . Let $ab \in I_2$ for some elements $a, b \in R_2$. If a or b is unit, we are done. So assume that a, b are nonunits in R_2 . Since $\phi_1(R_1)$ is not unique maximal ideal of R_1 , there exists a nonunit element $x \in R_1 - \phi_1(R_1)$. Then we have $(x, 1)(1, a)(1, b) = (x, ab) \in I - \phi(I)$. Since I is a ϕ -1-absorbing prime ideal of R, we have either $(x, 1)(1, a) = (x, a) \in I$ or $(1, b) \in I$ implying or $a \in I_2$ or $b \in I_2$. Therefore, I_2 is a prime ideal of R_2 .

Recall that a commutative ring R is said to be a *quasi-local* if it has a unique maximal ideal [16]. Otherwise, we say R is not quasi-local or non-quasi-local.

THEOREM 8. Let R_1, R_2 be commutative rings such that $\phi_i(I_i)$ is not unique maximal ideal of R_i (e.g. R_i is not quasi-local) and $\phi_i : \mathcal{I}(R_i) \to \mathcal{I}(R_i) \cup \{\emptyset\}$ for each i = 1, 2. Suppose that $I = I_1 \times I_2$ is nonzero ideal, where I_i is an ideal of R_i for each $i = 1, 2, \ \phi = \phi_1 \times \phi_2$ and $\phi(I) \neq I$. Then the following statements are equivalent.

(i) I is a ϕ -1-absorbing prime ideal of $R = R_1 \times R_2$.

(ii) $I = I_1 \times R_2$ for some prime ideal I_1 of R_1 and $I = R_1 \times I_2$ for some prime ideal I_2 of R_2 .

(iii) I is a prime ideal of R.

(iv) I is a weakly prime ideal of R.

(v) I is a 1-aborbing prime ideal of R.

PROOF. $(i) \Rightarrow (ii)$: Follows from Theorem 7.

 $(ii) \Rightarrow (iii)$: Clear.

 $(iii) \Leftrightarrow (iv)$: Follows from [2, Theorem 7].

 $(iii) \Rightarrow (v)$: Follows from [19, Definition 2.1].

 $(v) \Rightarrow (i)$: Follows from the fact that $\phi_{\emptyset} \leq \phi$ and Proposition 1.

THEOREM 9. Let R_1, R_2 be commutative rings such that $\phi_i(I_i)$ is not unique maximal ideal of R_i (e.g. R_i is not quasi-local) and $\phi_i : \mathcal{I}(R_i) \to \mathcal{I}(R_i) \cup \{\emptyset\}$ for each i = 1, 2, ..., n. Suppose that $I = I_1 \times I_2 \times \cdots \times I_n$ is nonzero ideal, where I_i is an ideal of R_i for each i = 1, 2, ..., n, $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_n$ and $\phi(I) \neq I$. Then the following statements are equivalent.

I is a ϕ -1-absorbing prime ideal of $R = R_1 \times R_2$.

(ii) $I = R_1 \times R_2 \times \cdots \times R_{t-1} \times I_t \times R_{t+1} \times \cdots \times R_n$ for some prime ideal I_t of R_t and $1 \le t \le n$.

(iii) I is a prime ideal of R.

(iv) I is a weakly prime ideal of R.

(v) I is a 1-absorbing prime ideal of R.

PROOF. We use induction on n. If n = 1, the claim is clear. If n = 2, the claim follows from Theorem 8. Now, assume that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ is true for all k < n. Let $I' = I_1 \times I_2 \times \cdots \times I_{n-1}$, $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$

and $\phi' = \phi_1 \times \phi_2 \times \cdots \times \phi_{n-1}$. Then note that $I = I' \times I_n$, $R = R' \times R_n$ and $\phi = \phi' \times \phi_n$. The rest follows from induction hypothesis and Theorem 8.

LEMMA 1. Let (R, m) be a quasi-local ring and $\mathfrak{m}^3 \subseteq \phi(I)$ for every proper ideal I of R. Then, every proper ideal of R is a ϕ -1-absorbing prime ideal.

PROOF. Let I be a nonzero proper ideal of R. Assume that I is not ϕ -1-absorbing prime ideal. Then, there exist nonunit elements $x, y, z \in R$ such that $xyz \in I - \phi(I)$ but $xy \notin I$ and $z \notin I$. Since x, y, z are nonunits, they are elements of \mathfrak{m} . So, $xyz \in \mathfrak{m}^3 \subseteq \phi(I)$, a contradiction.

A ring R is said to be an *indecomposable ring* if its all idempotents are 0 and 1. Otherwise, we say R is decomposable. It is well know that a ring R is decomposable if and only if $R = R_1 \times R_2$ for some commutative rings R_1 and R_2 .

Recall that a commutative ring R is said to be a von Neumann regular ring if its each ideal is idempotent, or equivalently, for each $x \in R$, there exists an idempotent element $e \in R$ such that (x) = (e) [18]. The concept of von Neumann regular rings and its generalizations have drawn considerable interest and have been widely studied by many authors. See, for example, [3], [11] and [12]. Now, in the following, we characterize all rings over which every proper ideal is almost 1-absorbing prime ideal.

THEOREM 10. Let R be a ring. Then every proper ideal is almost 1-absorbing prime if and only (R, \mathfrak{m}) is either quasi-local with $\mathfrak{m}^3 = (0)$ or R is a von Neumann regular ring.

PROOF. (\Leftarrow) : Suppose that (R, \mathfrak{m}) is quasi-local with $\mathfrak{m}^3 = (0)$. Then by previous Lemma, every ideal is almost 1-absorbing prime. If R is von Neumann regular ring, then every ideal is idempotent so that every ideal is almost 1-absorbing prime.

 (\Rightarrow) : Now, suppose that every proper ideal is almost 1-absorbing prime. First, we will show that $(a^3) = (a^4)$ for each element $a \in R$. If a is unit, then we are done. So assume that a is not unit. Take a maximal ideal \mathfrak{m} of R. If $a \notin \mathfrak{m}$, then $\frac{a}{1}$ is unit in $R_{\mathfrak{m}}$ so that we have $(a^3)_{\mathfrak{m}} = (a^4)_{\mathfrak{m}}$. So suppose that $a \in \mathfrak{m}$. Then by Theorem 6, every proper ideal of $R_{\mathfrak{m}}$ is almost 1-absorbing prime. Since $\frac{a^3}{1} \in (\frac{a^3}{1})$ and $(\frac{a^3}{1})$ is almost 1-absorbing prime, we have either $\frac{a^2}{1} \in (\frac{a^3}{1})$ or $\frac{a^3}{1} \in (\frac{a^3}{1})^2$, which implies that $(a^3)_{\mathfrak{m}} = (a^4)_{\mathfrak{m}}$. Since $(a^3)_{\mathfrak{m}} = (a^4)_{\mathfrak{m}}$ for each maximal ideal \mathfrak{m} of R, we have $(a^3) = (a^4)$ and thus $(a^3) = (a^3)^2$. This implies that $(a^3) = (e)$ for some idempotent $e \in R$. If R is not decomposable ring, then for each nonunit $a \in R$, $a^3 = (0)$ and this shows that (R,\mathfrak{m}) is quasi-local with $\mathfrak{m}^3 = (0)$, where $\mathfrak{m} = \sqrt{0}$. Now, suppose that $R = R_1 \times R_2$ for some commutative rings R_1 and R_2 . If R_1 is not von Neumann regular, then there exists an ideal I of R such that $I^2 \neq I$. Now take the ideal $J = I \times 0$ of R. Since J is almost 1-absorbing prime, by Theorem 7, $J = I \times 0 = I \times R_2$ which is a contradiction. Thus R_1 is a von Neumann regular ring. Similarly, R_2 is von Neumann regular ring and so is $R = R_1 \times R_2$. \Box

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